LUND'S COMPANION

TO

WOOD'S ALGEBRA.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

A

COMPANION TO WOOD'S ALGEBRA,

CONTAINING

SOLUTIONS OF VARIOUS QUESTIONS AND PROBLEMS IN ALGEBRA,

AND FORMING

A KEY

TO THE CHIEF DIFFICULTIES FOUND IN THE COLLECTION OF EXAMPLES APPENDED TO WOOD'S ALGEBRA.

THIRD EDITION.

BY

THOMAS LUND, B.D.

LATE FELLOW AND SADLERIAN LECTURER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONDON:
LONGMAN, GREEN, LONGMAN, AND ROBERTS.
1860.

ADVERTISEMENT TO THE THIRD EDITION.

SINCE the Second Edition of this work was issued a new Edition of *Wood's Algebra* has been called for, in which was inserted a collection of all the best questions and problems to be found in the Cambridge Examination Papers of *the preceding four years.* It became necessary, therefore, to adapt the *Companion* to this altered state of the *Algebra;* and, accordingly, a *Supplement* of 74 pages was printed, and appended to all the Copies then remaining in the hands of the publisher. It was also published separately at a low price, (and may still be had) to complete the copies of the *Companion* previously purchased.

This *Supplement* is now incorporated into the present Edition; and a further addition is made of the Equations and Algebraical Problems, proposed at the Examinations in St John's College during the last *two* years, 1858 and 1859, for the *Solutions* of which the Author is mainly indebted to the Rev. J. R. Lunn, Fellow and Sadlerian Lecturer of that College.

This work, it is to be observed, is more than a mere KEY for *Schoolmasters* and *Private Tutors.* Its chief use should be, to teach *Students* the best and neatest *modes of working*, as well as the application of numerous *artifices* known to the practised analyst, but not readily occurring to the minds of beginners. With this view every Solution is given at full length, and in the exact form suited to a Cambridge Examination.

It is also to be noticed, that each Example, or Problem, is here *enunciated* at the head of its Solution; so that the book, though a fitting *Companion* to Wood, is not inseparable from it, but may be used, as a *Book of Exercises*, with any other treatise on *Algebra.*

T. L.

MORTON RECTORY, near ALFRETON.
March 1, 1860.

CONTENTS

INTRODUCTION.

[N.B. The Articles and Examples referred to by number are those in Wood's *Algebra.*]

VULGAR FRACTIONS.

Ex. 12. Which is greater $\sqrt{\frac{2}{3}}$, or $\sqrt[3]{\frac{2}{3}}$?

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6},\text{ and }\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{18}{27}}=\frac{1}{3}\sqrt[3]{18},$$

$$\therefore\sqrt{\frac{2}{3}}>\text{ or }<\sqrt[3]{\frac{2}{3}},\text{ according as }\sqrt{6}>\text{ or }<\sqrt[3]{18}.$$

Now $\sqrt{6}=2{\cdot}4$ &c., and $\sqrt[3]{18}=2{\cdot}6$ &c.;

$$\therefore\sqrt[3]{\frac{2}{3}}>\sqrt{\frac{2}{3}}.$$

Ex. 23. Reduce $\dfrac{4\frac{1}{3}\times4\frac{1}{3}\times4\frac{1}{3}-1}{4\frac{1}{3}\times4\frac{1}{3}-1}$.

$$\text{Fraction }=4\tfrac{1}{3}+\frac{3\frac{1}{3}}{4\frac{1}{3}\times4\frac{1}{3}-1},$$

$$=4\tfrac{1}{3}+\frac{30}{13\times13-9},$$

$$=4\tfrac{1}{3}+\frac{30}{160},$$

$$=4+\frac{1}{3}+\frac{3}{16},$$

$$=4\tfrac{25}{48}.$$

Ex. 30. $\frac{2}{3}$ multiplied by 3 signifies $\frac{2}{3}$ *taken three times*, that is, $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$. What does $\frac{2}{3}$ multiplied by $\frac{3}{4}$ signify?

"Multiplication of one number by another is the taking as many of one number as there are units in the other. Thus to multiply 12 by 7 is to take as many twelves as there are units in 7, or to take 12 as many times as you must take 1 to make 7. Thus, what is done with 1 to make 7, is done with 12 to make 7 times 12. For example,

$$7 \text{ is } 1+1+1+1+1+1+1,$$

$$7 \text{ times } 12 \text{ is } 12+12+12+12+12+12+12.$$

When the same thing is done with two fractions, the result is still called multiplication. There is this difference, that whereas a whole number is made by adding 1 to itself a number of times, a fraction is made by dividing 1 into a number of equal parts, and adding one of these parts to itself a number of times." De Morgan's *Arithmetic*, Art. 118.

Hence to multiply (in this sense) $\frac{2}{3}$ by $\frac{3}{4}$, what is done with 1 to make $\frac{3}{4}$, the same must be done with $\frac{2}{3}$. But, to make $\frac{3}{4}$, 1 is divided into 4 equal parts, and 3 of them are taken. Therefore, to make $\frac{2}{3}$ multiplied by $\frac{3}{4}$, $\frac{2}{3}$ must be divided into 4 equal parts, and 3 of them must be taken. Now $\frac{2}{3}$ is the same as $\frac{8}{12}$, or $\frac{2}{12}+\frac{2}{12}+\frac{2}{12}+\frac{2}{12}$, (dividing it into 4 equal parts,) and three of these parts are together $\frac{6}{12}$, or $\frac{1}{2}$; that is $\frac{2}{3}\times\frac{3}{4}=\frac{6}{12}=\frac{1}{2}$.

Ex. 32. An article which cost 3*s*. 6*d*. is sold for 3*s*. $10\frac{1}{2}$*d*.; what is that per cent. profit?

The actual profit is $4\frac{1}{2}$*d*.; ∴ the profit *per cent.*, that is, for 100£'s worth of the article, will be as many times $4\frac{1}{2}$*d*. as 3*s*. 6*d*. is contained in £100,

or £100÷3*s*. 6*d*.×$4\frac{1}{2}$*d*., that is £$100\times\frac{4\frac{1}{2}}{42}$, or £$\frac{900}{84}$, or £$10\frac{5}{7}$.

Ex. 35. A shilling weighs 3 dwts. 15 grs. of which 3 parts out of 40 are alloy, and the rest pure silver. How much per cent. is there of alloy, and what weight of pure silver?

(1) Since 3 parts out of 40 are alloy,

37 .... are pure silver;

$$\therefore \text{ the weight of pure silver} = \frac{37}{40} \text{ of } \overset{\text{dwts.}}{3} \,.\, \overset{\text{grs.}}{15} = \frac{\overset{\text{dwts.}}{134} \,.\, \overset{\text{grs.}}{3}}{40},$$

$$= \overset{\text{dwts.}}{3} \,.\, \overset{\text{grs.}}{8\tfrac{19}{40}}.$$

(2) There is *one* part of alloy for every $\frac{40}{3}$ parts;

$\therefore$ the number of parts of alloy out of 100 parts will be the number of times $\frac{40}{3}$ is contained in 100, that is,

$$\text{the alloy } \textit{per cent.} = \frac{100}{\frac{40}{3}} = \frac{300}{40} = 7\tfrac{1}{2}.$$

Ex. 39. Divide $\frac{3}{5}$ into two parts, so that one is greater than the other by $\frac{4}{13}$.

$$\frac{1}{2} \text{ of } \frac{3}{5} \text{ is } \frac{3}{10}, \text{ and } \frac{1}{2} \text{ of } \frac{4}{13} \text{ is } \frac{2}{13};$$

$$\therefore \text{ one part is } \frac{3}{10} + \frac{2}{13}, \text{ or } \frac{59}{130};$$

$$\text{and the other is } \frac{3}{10} - \frac{2}{13}, \text{ or } \frac{19}{130}.$$

DECIMAL FRACTIONS.

Ex. 16. Find $\frac{1}{\sqrt{3}}$ correct to 7 places of decimals; and $\sqrt{\frac{5{\cdot}04}{0{\cdot}012}}$ to two places.

(1) Since $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{1}{3}\sqrt{3}$, extract the square root of 3 to 7 places of decimals, and take the 3rd part of the root.

(2) Since $\frac{5{\cdot}04}{0{\cdot}012} = \frac{0{\cdot}42}{0{\cdot}001}$, (dividing num. and denom. by 12),

$= \frac{420}{1}$, (multiplying................by 1000),

$\therefore$ root required $= \sqrt{420} = 20{\cdot}49$.

Ex. 19. Express the following in a decimal of 4 places:—

$$16\times\left\{\frac{1}{5}-\frac{1}{3\times5^3}+\frac{1}{5\times5^5}-\frac{1}{7\times5^7}+\&c.\right\}-\frac{4}{239}.$$

$$\frac{1}{5}=0{\cdot}2,$$

$$\frac{1}{5^3}=\frac{2^3}{10^3}=0{\cdot}008, \qquad \therefore \frac{1}{3}\times\frac{1}{5^3}=0{\cdot}002666\ \ldots,$$

$$\frac{1}{5^5}=\frac{2^5}{10^5}=0{\cdot}00032, \qquad \therefore \frac{1}{5}\times\frac{1}{5^5}=0{\cdot}000064$$

$$\frac{1}{5^7}=\frac{2^7}{10^7}=0{\cdot}0000128, \qquad \therefore \frac{1}{7}\times\frac{1}{5^7}=0{\cdot}0000018\ldots$$

$$\text{Also } \frac{4}{239}=0{\cdot}01673,$$

$$\begin{aligned}\therefore \text{Ans.} &=16\times\{0{\cdot}20006-0{\cdot}00266\}-0{\cdot}01673,\\ &=16\times0{\cdot}1974-0{\cdot}01673,\\ &=3{\cdot}1584-0{\cdot}01673,\\ &=3{\cdot}1416.\end{aligned}$$

Ex. 20. Express a degree ($69\frac{1}{2}$ miles) in metres, 32 metres being equal to 35 yards nearly.

$$\begin{aligned}69\tfrac{1}{2}\text{ miles} &=1760\times69\tfrac{1}{2}\text{ yards},\\ &=\frac{1760\times69\frac{1}{2}}{35}\times32\text{ metres},\\ &=\frac{352\times69\frac{1}{2}\times32}{7}\text{ metres},\\ &=\frac{782848}{7}=111835{\cdot}42857\text{ metres}.\end{aligned}$$

Ex. 21. The true length of a year is $365{\cdot}24224$ days. Find what the error amounts to by the common reckoning in 4 centuries.

Assuming every fourth year to be leap year, the common year will average $365\frac{1}{4}$ days, or $365{\cdot}25$ days;

$$\begin{aligned}\therefore \text{error per year} &=365{\cdot}25-365{\cdot}24224,\\ &=0{\cdot}00776\text{ days};\\ \text{and error in 400 years} &=0{\cdot}00776\times400,\\ &=3{\cdot}104\text{ days}.\end{aligned}$$

But in 4 centuries 3 intercalary days are omitted;

$\therefore$ error required $=0{\cdot}104$ days.

Ex. 22. A square inch plate of metal of 0·05 inches thickness is drawn into a wire of uniform thickness 50 feet long; find the thickness of the wire.

The quantity of metal employed in cubic inches = the surface in square inches × the thickness in linear inches,

$$=1\times0{\cdot}05=0{\cdot}05.$$

$$\left.\begin{array}{c}\text{And the thickness or section of}\\ \text{the wire in square inches}\end{array}\right\}=\frac{\text{number of cubic inches of metal}}{\text{length of wire in inches}},$$

$$=\frac{0{\cdot}05}{600}=0{\cdot}0000833\ldots..$$

Ex. 27. Two distances are measured in inches, and are known to be correct within a quarter of a hundredth of an inch each way, being 11·87 and 9·95. How far can their *product* be depended upon for accuracy?

The product of the two given numbers is 118·1065. But

11·87 may be any thing between $11{\cdot}87+\frac{1}{4}$ of $\frac{1}{100}$,

and $11{\cdot}87-\frac{1}{4}$ of 100 inclusive,

.... $11{\cdot}87+0{\cdot}0025$, and $11{\cdot}87-0{\cdot}0025$

.... 11·8725, and 11·8675.

Similarly,

9·95 .... 9·9525, and 9·9475.

So that the correct product may be as great as $11{\cdot}8725\times9{\cdot}9525$, or may be as small as $11{\cdot}8675\times9{\cdot}9475$,

the former of which gives 118·16105625,

and the latter .... 118·05195625.

Therefore the product of the given numbers may not be correct beyond the *integral part*.

ALGEBRA.

MULTIPLICATION.

Ex. 34. Find the coefficient of x^4 in the product of x^2+px+q and $x^4-ax^3+bx^2-cx+d$.

Whole quantity in the *product* involving x^4

$$=x^2.bx^2-px.ax^3+qx^4,$$
$$=bx^4-apx^4+qx^4=(b-ap+q)x^4,$$

$\therefore$ coefficient required $=b-ap+q$.

Ex. 38. Simplify

$$\frac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\frac{2}{3}(x-1)x(x+1).$$

$$\text{Ans.} =x\left[\frac{1}{6}\{(x+1)(x+2)+(x-1)(x-2)\}+\frac{2}{3}(x-1)(x+1)\right],$$

$$=x\left[\frac{1}{6}\{x^2+3x+2+x^2-3x+2\}+\frac{2}{3}(x^2-1)\right],$$

$$=x\left[\frac{x^2}{3}+\frac{2}{3}+\frac{2x^2}{3}-\frac{2}{3}\right],$$

$$=x.\frac{3x^2}{3}=x^3.$$

Ex. 39. Prove that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)$ is equal to $(a-b)(b-c)(a-c)$.

$$\text{Product}=\left.\begin{array}{l}(a-b)\{x^2-(a+b)x+ab\}\\ +(b-c)\{x^2-(b+c)x+bc\}\\ +(c-a)\{x^2-(c+a)x+ac\}\end{array}\right\},$$

$$=0\times x^2-\left\{\begin{array}{l}(a^2-b^2)x+(a-b)ab\\ (b^2-c^2)x+(b-c)bc\\ (c^2-a^2)x+(c-a)ac,\end{array}\right.$$

$$=(a-b)ab+(b-c)bc+(c-a)ac,$$
$$=(a-b)(ab-ac-bc+c^2),$$
$$=(a-b)(b-c)(a-c).$$

Ex. 40. Find the difference between $a(b+c)^2+b(a+c)^2+c(a+b)^2$ and $(a+b)(a-c)(b-c)+(a-b)(a-c)(b+c)-(a-b)(b-c)(a+c)$.

1st quantity $=ab^2+ba^2+ac^2+ca^2+bc^2+cb^2+6abc$.

$$
\begin{aligned}
\text{2nd quantity} &= \{a^2+(b-c)a-bc\}(b-c) \\
&\quad +(a-b)(ab+ac-bc-c^2-ab+ac-bc+c^2), \\
&= a^2b-a^2c+ab^2+ac^2-2abc-b^2c+bc^2+2a^2c-4abc+2b^2c, \\
&= ab^2+ba^2+ac^2+ca^2+bc^2+cb^2-6abc,
\end{aligned}
$$

$\therefore$ Diff. req. $=12abc$.

GREATEST COMMON MEASURE.

Ex. 22. Find the G.C.M. of $a^2+2b^2+(a+2b)\sqrt{ab}$ and $a^2-b^2+(a-b)\sqrt{ab}$.

$$
\begin{array}{r|l}
a^2-b^2+(a-b)\sqrt{ab} & a^2+2b^2+(a-2b)\sqrt{ab} \;\big(1 \\
 & \underline{a^2-b^2+(a-b)\sqrt{ab}} \\
 & \qquad 3b^2+3b\sqrt{ab} \\
 & \qquad =3b\sqrt{b}(\sqrt{a}+\sqrt{b}),
\end{array}
$$

$$
\begin{array}{r|l}
\sqrt{a}+\sqrt{b} & a^2+a\sqrt{ab}-b\sqrt{ab}-b^2 \;\big(a\sqrt{a}-b\sqrt{b} \\
 & \underline{a^2+a\sqrt{ab}} \\
 & \qquad -b\sqrt{ab}-b^2 \\
 & \qquad \underline{-b\sqrt{ab}-b^2} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore$ G.C.M. $=\sqrt{a}+\sqrt{b}$.

Or, as follows,

$$
\begin{aligned}
\text{1st quantity} &= a^2+a\sqrt{ab}+2b\sqrt{ab}+2b^2, \\
&= a\sqrt{a}(\sqrt{a}+\sqrt{b})+2b\sqrt{b}(\sqrt{a}+\sqrt{b}), \\
&= (a\sqrt{a}+2b\sqrt{b})(\sqrt{a}+\sqrt{b}). \\
\text{2nd quantity} &= a^2+a\sqrt{ab}-b\sqrt{ab}-b^2, \\
&= a\sqrt{a}(\sqrt{a}+\sqrt{b})-b\sqrt{b}(\sqrt{a}+\sqrt{b}), \\
&= (a\sqrt{a}-b\sqrt{b})(\sqrt{a}+\sqrt{b}).
\end{aligned}
$$

And, since $a\sqrt{a}+2b\sqrt{b}$ and $a\sqrt{a}-b\sqrt{b}$, have no common factor greater than 1, $\therefore$ $\sqrt{a}+\sqrt{b}$ is the Greatest Common Measure required.

Ex. 23. Find the G.C.M. of $\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1$ and $x^2-\frac{x}{4}-\frac{1}{4}$.

$$\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1\,\Big)\,x^2-\frac{x}{4}-\frac{1}{4}\,\Big(\,3$$

$$x^2+\frac{11x}{2}\sqrt{x+1}-3x-3$$

$$-\frac{11x}{2}\sqrt{x+1}+\frac{11x}{4}+\frac{11}{4}$$

$$=-\frac{11}{2}\sqrt{x+1}.\{x-\tfrac{1}{2}\sqrt{x+1}\}$$

$$\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1$$
$$6$$

$$x-\tfrac{1}{2}\sqrt{x+1}\,\Big)\,2x^2+11x\sqrt{x+1}-6x-6\,\Big(\,2x+12\sqrt{x+1}$$
$$2x^2-x\sqrt{x+1}$$
$$12x\sqrt{x+1}-6x-6$$
$$12x\sqrt{x+1}-6x-6$$
$$0$$

$\therefore\ x-\frac{1}{2}\sqrt{x+1}$ is the G. C.M. required.

Ex. 25. Find the G. C. M. of $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$ and $2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2$.

$$\begin{aligned}
\text{1st quantity} &= x^3(a^2-2ab+b^2)+a^3(a^2-2ab+b^2),\\
&=(a-b)^2(x^3+a^3),\\
&=(a-b)^2(x+a)(x^2-ax+a^2)\ldots\ldots\ldots\ldots\ldots\ldots(1).\\
\text{2nd quantity} &= 2x^4(a^2-b^2)-5a^2x^2(a^2-b^2)+3a^4(a^2-b^2),\\
&=(a^2-b^2)(2x^4-5a^2x^2+3a^4),\\
&=(a^2-b^2)\{2x^2(x^2-a^2)-3a^2(x^2-a^2)\},\\
&=(a^2-b^2)(x^2-a^2)(2x^2-3a^2),\\
&=(a-b)(a+b)(x-a)(x+a)(2x^2-3a^2)\ldots\ldots(2).
\end{aligned}$$

Now the only common factors in (1) and (2) are $a-b$, and $x+a$; therefore the G.C.M. required is $(a-b)(x+a)$.

Ex. 26. Find the G.C.M. of

$$2(y^3-2y^2-y+2)x^3+3(y^2-1)x^2-(2y^3-y^2-2y+1)$$

and $3(y^3-4y^2+5y-2)x^2+7(y^2-2y+1)x-(3y^3-5y^2+y+1)$.

1st quantity

$$\begin{aligned}
&=2\{y^2(y-2)-(y-2)\}x^3+3(y^2-1)x^2-\{y^2(2y-1)-(2y-1)\},\\
&=(y^2-1)\{2(y-2)x^3+3x^2-2y+1\},\\
&=(y^2-1)\{2y(x^3-1)-4x^2(x-1)-(x^2-1)\},\\
&=(x-1)(y^2-1)\{2y(x^2+x+1)-4x^2-x-1\}\ldots\ldots\ldots\ldots(1).
\end{aligned}$$

2nd quantity

$$\begin{aligned}
&=3\{y(y^2-2y+1)-2(y^2-2y+1)\}x^2+7(y-1)^2x-3y(y^2-2y+1)-(y^2-2y+1),\\
&=(y-1)^2\{(3y-6)x^2+7x-3y-1\},\\
&=(y-1)^2\{3y(x^2-1)-6x(x-1)+x-1\},\\
&=(x-1)(y-1)^2\{3y(x+1)-6x+1\}\ldots\ldots\ldots\ldots(2).
\end{aligned}$$

Now the latter factors of (1) and (2) cannot be further developed, therefore the G.C.M. required is

$$(x-1)(y-1), \text{ or } xy-x-y+1.$$

Ex. 27. Find the value of y which will make

$2(y^2+y)x^2+(11y-2)x+4$ and $2(y^3+y^2)x^3+(11y^2-2y)x^2+(y^2+5y)x+5y-1$ have a Common Measure.

$$2(y^2+y)x^2+(11y-2)x+4\ \big)\ 2(y^3+y^2)x^3+(11y^2-2y)x^2+(y^2+5y)x+5y-1\ \big(\ yx$$

$$\begin{array}{r}
2(y^3+y^2)x^3+(11y^2-2y)x^2+4yx\\
\hline
(y^2+y)x+5y-1
\end{array}$$

$$(y^2+y)x+5y-1\ \big)\ 2(y^2+y)x^2+(11y-2)x+4\ \big(\ 2x$$

$$\begin{array}{r}
2(y^2+y)x^2+(10y-2)x\\
\hline
yx+4
\end{array}$$

$$yx+4\ \big)\ (y^2+y)x+5y-1\ \big(\ y+1$$

$$\begin{array}{r}
(y^2+y)x+4y+4\\
\hline
y-5
\end{array}$$

Hence, if $y-5=0$, that is, if $y=5$, the proposed quantities have a Common Measure, viz. $5x+4$.

Ex. 28. Find the G.C.M. of $6ax^2-6ab^2-54a-21x^2+21b^2+189$, $6ax^5+60a-21x^5-210$, and $6ab^3-12a-21b^3+42$.

1st quantity $=(6a-21)x^2-(6a-21)b^2-9(6a-21)$,

$=(6a-21)(x^2-b^2-9)\ldots\ldots(1)$.

2nd quantity $=(6a-21)(x^2+10)\ldots\ldots\ldots(2)$.

3rd quantity $=(6a-21)(b^2-2)\ldots\ldots\ldots(3)$.

Hence, the latter factors in (1), (2), (3) having no Common Measure greater than 1, the G.C.M. required is $6a-21$.

Ex. 29. Find the G.C.M. of $6a^3x^2+4ax^3-10axy-3a^2xy-2x^2y+5y^2$, $10a^2x^2-2ax^3y-6axy^2-5axy+x^2y^2+3y^3$, and $-4ax^2-6a^2bxy+2ab^2x^2y-2abx+2xy+3aby^2-b^2xy^2+by$.

1st quantity $=2ax(3a^2x+2x^2)-y(3a^2x+2x^2)-5y(2ax-y)$,

$=(2ax-y)(3a^2x+2x^2-5y)\ldots\ldots\ldots\ldots\ldots(1)$.

2nd quantity $=2ax(5ax-3y^2)-(5ax-3y^2)y-(2ax-y)x^2y$,

$=(2ax-y)(5ax-3y^2-x^2y)\ldots\ldots\ldots\ldots\ldots(2)$.

3rd quantity

$=-2ax(2x+b)+y(2x+b)-2ax(3aby-b^2xy)+y(3aby-b^2xy)$,

$=-(2ax-y)(2x+b+3aby-b^2xy)\ldots\ldots\ldots\ldots\ldots(3)$.

Hence, the latter factors in (1), (2) and (3) having no Common Measure greater than 1, the G.C.M. required is

$-(2ax-y)$, that is, $y-2ax$.

Ex. 30. If $x+a$ be the Greatest Common Measure of x^2+px+q, and $x^2+p'x+q'$, shew that $a=\dfrac{q-q'}{p-p'}$.

Each of the quantities x^2+px+q, and $x^2+p'x+q'$, is divisible by $x+a$ without remainder; $\therefore$ performing the division thus,

$$
\begin{array}{l}
x+a)\ x^2+px+q\ (x+p-a \\
\qquad\ \ \underline{x^2+ax\qquad\quad} \\
\qquad\qquad (p-a)x+q \\
\qquad\qquad \underline{(p-a)x+(p-a)a} \\
\text{Rem}^{\text{r}}.=\ q-(p-a)a=0. \\
\text{Sim}^{\text{y}}.\qquad q'-(p'-a)a=0,
\end{array}
$$

$$\therefore\ q-q-(p-p)a=0,$$

$$\text{or } a=\frac{q-q'}{p-p'}.$$

Ex. 31. Shew that x^2+qx+1, and x^3+px^2+qx+1, have a common factor of the form $x+a$, when $(p-1)^2-q(p-1)+1=0$.

Since one factor of x^2+qx+1 is $x+a$, the other must be $x+\frac{1}{a}$, to make the last term 1,

$$\therefore\ q=a+\frac{1}{a}.$$

Again, $x^3+px^2+qx+1=x^3+(p-1)x^2+x^2+qx+1$, and is divisible by $x+a$; but x^2+qx+1 is divisible by $x+a$; $\therefore$ also $x^3+(p-1)x^2$, or $x^2(x+\overline{p-1})$ is divisible by $x+a$; that is, $p-1=a$;

$$\therefore\ q=p-1+\frac{1}{p-1},$$

$$\text{or } (p-1)^2-q(p-1)+1=0.$$

Another Method. By NOTE 1, *Alg.*, since each quantity is divisible by $x+a$,

$$a^2-qa+1=0, \text{ and } a^3+pa^2-qa+1=0;$$

$$\therefore \text{ by subtraction, } a^3-(p-1)a^2=0,$$

$$\text{or } a=p-1;$$

$$\therefore\ (p-1)^2-q(p-1)+1=0.$$

LEAST COMMON MULTIPLE.

Ex. 10. Find the Least Common Multiple of

$$3x^2-11x+6,\quad 2x^2-7x+3,\quad \text{and } 6x^2-7x+2.$$

$$3x^2-11x+6=3x(x-3)-2(x-3)=(3x-2)(x-3)\ldots\ldots\ldots(1).$$

$$2x^2-7x+3=2x(x-3)-(x-3)=(2x-1)(x-3)\ldots\ldots\ldots(2).$$

$$6x^2-7x+2=3x(2x-1)-2(2x-1)=(3x-2)(2x-1)\ldots\ldots\ldots(3).$$

Now of (1) and (2) $x-3$ is the Greatest Common Measure;
$\therefore$ the Least Common Multiple of (1) and (2) is

$$(3x-2)(2x-1)(x-3)\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, the Greatest Common Measure of (3) and (4) is

$$(3x-2)(2x-1),$$

$\therefore$ the Least Common Multiple required is

$$(3x-2)(2x-1)(x-3),$$
$$=(6x^2-7x+2)(x-3),$$
$$=6x^3-25x^2+23x-6.$$

Ex. 12. Find the Least Common Multiple of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4-2x^3+2x^2-2x+1$

$$x^3-3x^2+3x-1=(x-1)^3 \text{.......................................(1).}$$
$$x^3-x^2-x+1=x^2(x-1)-(x-1)=(x^2-1)(x-1),$$
$$=(x-1)^2(x+1) \text{................(2).}$$
$$x^4-2x^3+2x-1=x^2(x^2-2x+1)-(x^2-2x+1)=(x-1)^2(x^2-1),$$
$$=(x-1)^3(x+1) \text{........(3).}$$
$$x^4-2x^3+2x^2-2x+1=x^2(x^2-2x+1)+x^2-2x+1=(x-1)^2(x^2+1) \text{........(4).}$$

Now the Greatest Common Measure of (1) and (3) is $(x-1)^3$, $\therefore$ their Least Common Multiple is

$$(x-1)^3(x+1) \text{..................................(5).}$$

The Greatest Common Measure of (2) and (4) is $(x-1)^2$, $\therefore$ their Least Common Multiple is

$$(x-1)^2(x+1)(x^2+1) \text{..........................(6).}$$

The Greatest Common Measure of (5) and (6) is

$$(x-1)^2(x+1),$$

$\therefore$ the Least Common Multiple required is

$$(x-1)^3(x+1)(x^2+1),$$
$$=(x^4-2x^3+2x-1)(x^2+1),$$
$$=x^6-2x^5+x^4-x^2+2x-1.$$

Ex. 13. Shew that if $x+c$ be the Greatest Common Measure of x^2+ax+b, and $x^2+a'x+b'$, their Least Common Multiple will be $x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c$.

The Least Common Multiple required

$$=\frac{x^2+ax+b}{x+c}\cdot(x^2+a'x+b'),$$

$$x+c \,\big)\, x^2+ax+b \,\big(\, x+a-c$$
$$\underline{x^2+cx}$$
$$(a-c)x+b$$
$$\underline{(a-c)x+(a-c)c}$$
$$b-(a-c)c$$

Since x^2+ax+b is divisible by $x+c$, the remainder

$$b-(a-c)c=0.$$

Also, $b'-(a'-c)c=0$, since $x^2+a'x+b'$ is divisible by $x+c$.

$\therefore$ Least Common Multiple

$$=(x+a-c)(x^2+a'x+b'),$$
$$=x^3+(a+a'-c)x^2+\{b'+(a-c)a'\}x+(a-c)b',$$
$$=x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c.$$

FRACTIONS.

Ex. 22. Reduce $\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)(2x+1)}{1\times2\times3}$.

$$\text{Ans.}=\frac{x(x+1)}{3}\left\{x+2-\frac{2x+1}{2}\right\},$$
$$=\frac{x(x+1)}{3}\left\{x+2-x-\frac{1}{2}\right\},$$
$$=\frac{x(x+1)}{3}\cdot\frac{3}{2},$$
$$=\frac{x(x+1)}{2}.$$

Ex. 27. Reduce $\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}$.

$$\text{Ans.}=\frac{x^{3n}-1}{x^n-1}-\frac{x^{2n}-1}{x^n+1},$$
$$=x^{2n}+x^n+1-(x^n-1),$$
$$=x^{2n}+2.$$

Ex. 28. Reduce $\frac{a^2+b^2+ab}{(a^3-b^3)(x^{\frac{1}{2}}-a^{\frac{1}{2}})}-\frac{x^{\frac{1}{2}}}{(a-b)(x-a)}$.

Since $a^3-b^3=(a-b)(a^2+b^2+ab)$,

$$\text{Ans.}=\frac{1}{(a-b)(x^{\frac{1}{2}}-a^{\frac{1}{2}})}-\frac{x^{\frac{1}{2}}}{(a-b)(x-a)},$$
$$=\frac{x^{\frac{1}{2}}+a^{\frac{1}{2}}}{(a-b)(x-a)}-\frac{x^{\frac{1}{2}}}{(a-b)(x-a)},$$
$$=\frac{a^{\frac{1}{2}}}{(a-b)(x-a)}.$$

Ex. 30. Reduce

$$\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}.$$

$$\text{Ans.}=\frac{(b-c)(x+b)(x+c)-(a-c)(x+a)(x+c)+(a-b)(x+a)(x+b)}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)};$$

of which the Numerator

$$=(b-c)\{x^2+(b+c)\,x+bc\}-(a-c)\{x^2+(a+c)\,x+ac\}$$
$$+(a-b)\{x^2+(a+b)x+ab\},$$
$$=\{b-c-(a-c)+a-b\}\,x^2+(b^2-c^2-a^2+c^2+a^2-b^2)x+(b-c)bc$$
$$-(a-c)\,ac+(a-b)\,ab,$$
$$=(a-b)ab-(a-c)ac+(b-c)bc,$$
$$=a(ab-ac+c^2)-b(ab-bc+c^2),$$
$$=a(ab-ac-bc+c^2)-b(ab-ac-bc+c^2),$$
$$=(a-b)(ab-ac-bc+c^2),$$
$$=(a-b)(a-c)(b-c),$$

$$\therefore\ \text{Ans.}=\frac{1}{(x+a)(x+b)(x+c)}.$$

Ex. 31. Reduce

$$bc.\frac{a+d}{(a-b)(a-c)}+ac.\frac{b+d}{(b-a)(b-c)}+ab.\frac{c+d}{(c-a)(c-b)}.$$

$$\text{Ans.}=bc.\frac{a+d}{(a-b)(a-c)}-ac.\frac{b+d}{(a-b)(b-c)}+ab.\frac{c+d}{(a-c)(b-c)},$$

$$=\frac{bc(a+d)(b-c)-ac(b+d)(a-c)+ab(c+d)(a-b)}{(a-b)(b-c)(a-c)};$$

of which the Numerator

$$=bc(ab-ac+bd-cd)-ac(ab-bc+ad-cd)+ab(ac-bc+ad-bd),$$
$$=ab^2c-abc^2+b^2cd-bc^2d-a^2bc+abc^2-a^2cd+ac^2d+a^2bc-ab^2c+a^2bd-ab^2d,$$
$$=b^2cd-bc^2d-a^2cd+ac^2d+a^2bd-ab^2d,$$
$$=d(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2).$$

And the Denom^r. $=(a-b)(ab-bc-ac+c^2)$,

$$=a^2b-abc-a^2c+ac^2-ab^2+b^2c+abc-bc^2,$$
$$=b^2c-bc^2-a^2c+ac^2+a^2b-ab^2,$$

$\therefore$ Ans. $=d$.

Ex. 32. Reduce $\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)$

$$+\frac{a+c}{ac}(a^2+c^2-b^2).$$

$$\text{1st fraction}=\left(\frac{1}{a}+\frac{1}{b}\right)(a^2+b^2-c^2)=a+\frac{b^2}{a}-\frac{c^2}{a}+\frac{a^2}{b}+b-\frac{c^2}{b},$$

$$\text{2nd} \ldots\ldots =\left(\frac{1}{b}+\frac{1}{c}\right)(b^2+c^2-a^2)=b+\frac{c^2}{b}-\frac{a^2}{b}+\frac{b^2}{c}+c-\frac{a^2}{c},$$

$$\text{3rd} \ldots\ldots =\left(\frac{1}{a}+\frac{1}{c}\right)(a^2+c^2-b^2)=a+\frac{c^2}{a}-\frac{b^2}{a}+\frac{a^2}{c}+c-\frac{b^2}{c}.$$

And the sum $=2a+2b+2c=2(a+b+c)$.

Ex. 33. Reduce

$$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(c-a)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}.$$

$$\text{1st fraction}=\frac{\{a+(b-c)\}\{a-(b-c)\}}{(a+c+b)(a+c-b)}=\frac{a+b-c}{a+b+c},$$

$$\text{2nd} \ldots\ldots =\frac{\{b+(c-a)\}\{b-(c-a)\}}{(a+b+c)(a+b-c)}=\frac{b+c-a}{a+b+c},$$

$$\text{3rd} \ldots\ldots =\frac{\{c+(a-b)\}\{c-(a-b)\}}{(b+c+a)(b+c-a)}=\frac{c+a-b}{a+b+c},$$

$$\therefore \text{Sum}=\frac{a+b+c}{a+b+c}=1.$$

Ex. 34. Reduce $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$.

$$\text{Ans.}=\frac{1}{x-1+\dfrac{1}{\dfrac{4-x+x}{4-x}}}=\frac{1}{x-1+\dfrac{4-x}{4}},$$

$$=\frac{1}{\dfrac{4x-4+4-x}{4}}=\frac{4}{3x}.$$

Ex. 51. Reduce to lowest terms

$$\frac{a^2(b^2-c^2)-ab(2b^2+bc-c^2)+b^3(b+c)}{a^3(b^2+2bc+c^2)-a^2b(2b^2+3bc+c^2)+ab^3(b+c)}.$$

$$\text{Fraction}=\frac{a^2(b-c)-ab(2b-c)+b^3}{a^3(b+c)-a^2b(2b+c)+ab^3},\ (\text{divid}^{g}.\text{ by } b+c),$$

$$=\frac{b(a-b)^2-ac(a-b)}{ab(a-b)^2+a^2c(a-b)},$$

$$=\frac{b(a-b)-ac}{ab(a-b)+a^2c},$$

$$=\frac{a(b-c)-b^2}{a^2(b+c)-ab^2},\ \text{or}\ \frac{1}{a}\cdot\frac{a(b-c)-b^2}{a(b+c)-b^2}.$$

Ex. 52. Reduce to lowest terms

$$\frac{(c-d)a^2+6(bc-bd)a+9(b^2c-b^2d)}{(bc-bd+c^2-cd)a+3(b^2c+bc^2-b^2d-bcd)}.$$

$$\text{Num}^{r}. =(c-d)a^2+6b(c-d)a+9b^2(c-d),$$

$$\text{Denom}^{r}. =\{b(c-d)+c(c-d)\}a+3\{b^2(c-d)+bc(c-d)\},$$

$$\therefore\ \text{Fraction}=\frac{a^2+6ab+9b^2}{(b+c)a+3(b^2+bc)},\ \text{divid}^{g}.\text{ by } c-d,$$

$$=\frac{a(a+3b)+3b(a+3b)}{(b+c)(a+3b)},$$

$$=\frac{a+3b}{b+c}.$$

Ex. 55. Reduce $\dfrac{x-4-3x^{\frac{1}{2}}+4y^{\frac{1}{2}}-(xy)^{\frac{1}{2}}}{x-8-2x^{\frac{1}{2}}+12y^{\frac{1}{2}}-3(xy)^{\frac{1}{2}}}$.

$$\text{Ans.}=\frac{x^{\frac{1}{2}}(x^{\frac{1}{2}}-4)+x^{\frac{1}{2}}-4-y^{\frac{1}{2}}(x^{\frac{1}{2}}-4)}{x^{\frac{1}{2}}(x^{\frac{1}{2}}-4)+2(x^{\frac{1}{2}}-4)-3y^{\frac{1}{2}}(x^{\frac{1}{2}}-4)},$$

$$=\frac{(x^{\frac{1}{2}}-4)(x^{\frac{1}{2}}-y^{\frac{1}{2}}+1)}{(x^{\frac{1}{2}}-4)(x^{\frac{1}{2}}-3y^{\frac{1}{2}}+2)},$$

$$=\frac{x^{\frac{1}{2}}-y^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-3y^{\frac{1}{2}}+2}.$$

Ex. 4. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$.

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=\frac{2x^2-8ab}{x^2-2(a+b)x+4ab},$$

and if $x=\frac{4ab}{a+b}$, $(a+b)x=4ab$;

$$\therefore \text{ Ans.} =\frac{2x^2-8ab}{x^2-4ab}=2.$$

Ex. 5. Find the value of $\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx}$, when $x=\frac{a^n+b^n}{2}$.

$$2x=a^n+b^n, \quad \therefore 2nx=na^n+nb^n,$$

$$\therefore \text{ Value req}^{\text{d}}. =\frac{a^n}{na^n-nb^n}+\frac{b^n}{nb^n-na^n},$$

$$=\frac{a^n}{na^n-nb^n}-\frac{b^n}{na^n-nb^n},$$

$$=\frac{a^n-b^n}{n(a^n-b^n)}=\frac{1}{n}.$$

INVOLUTION AND EVOLUTION.

Ex. 19. Prove that $\left(x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\ldots\right)^2+\left(1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\ldots\right)^2$ is equal to 1.

By rule in Art. 142,

$$\text{1st square} = x^2-\frac{x^4}{3}+\frac{x^6}{3\times4\times5}-\ldots\ldots$$
$$+\frac{x^6}{2^2\times3^2}-\ldots\ldots$$

$$\text{2nd square} =1-x^2+\frac{x^4}{3\times4}-\frac{x^6}{3\times4\times5\times6}+\ldots\ldots$$
$$+\frac{x^4}{4}-\frac{x^6}{2\times3\times4}+\ldots\ldots$$

$\therefore$ sum

$$=1+\left\{\frac{1}{3\times4}+\frac{1}{4}-\frac{1}{3}\right\}x^4+\left\{\frac{1}{3\times4\times5}+\frac{1}{2^2\times3^2}-\frac{1}{3\times4\times5\times6}-\frac{1}{2\times3\times4}\right\}x^6+\ldots$$
$$=1+0+0+\ldots\ldots$$
$$=1.$$

Ex. 45. Find the square root (without the aid of the common rule) of $4\{(a^2-b^2)cd+(c^2-d^2)ab\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2$.

Proposed quantity

$$=4(a^2-b^2)^2c^2d^2+4(c^2-d^2)^2a^2b^2+8(a^2-b^2)(c^2-d^2)abcd$$
$$+(a^2-b^2)^2(c^2-d^2)^2-8(a^2-b^2)(c^2-d^2)abcd+16a^2b^2c^2d^2,$$
$$=(a^2-b^2)^2\{(c^2-d^2)^2+4c^2d^2\}+4a^2b^2\{(c^2-d^2)^2+4c^2d^2\},$$
$$=\{(a^2-b^2)^2+4a^2b^2\}\{(c^2-d^2)^2+4c^2d^2\},$$
$$=\{a^4+2a^2b^2+b^4\}\{c^4+2c^2d^2+d^4\},$$
$$=(a^2+b^2)^2(c^2+d^2)^2;$$

$\therefore$ square root required $=(a^2+b^2)(c^2+d^2)$.

Ex. 47. Find the coefficient of x^3 in the square of $(x+a)(x+b)(x+c)$.

$$\{(x+a)(x+b)(x+c)\}^2=\{x^3+(a+b+c)x^2+(ab+ac+bc)x+abc\}$$
$$\times\{x^3+(a+b+c)x^2+(ab+ac+bc)x+abc\},$$

and the coefficient of x^3 in this product

$$=abc(a+b+c)+(ab+ac+bc)^2+abc(a+b+c),$$
$$=a^2b^2+a^2c^2+b^2c^2+4abc(a+b+c).$$

Ex. 2. Shew that the sum of the cubes of any three consecutive integers is divisible by three times the second of them.

Let $a-1$, a, $a+1$, be the 3 integers, then

$$(a-1)^3+a^3+(a+1)^3=a^3-3a^2+3a-1+a^3+a^3+3a^2+3a+1,$$
$$=3a^3+6a,$$
$$=3a(a^2+2), \text{ which is divisible by } 3a.$$

Ex. 4. Prove that $x^4+px^3+qx^2+rx+s$ is a perfect square, if $p^2s=r^2$, and $q=\frac{p^2}{4}+2\sqrt{s}$.

Assuming $x^4+px^3+qx^2+rx+s$ to be a perfect square, let it be represented by $(x^2+ax+b)^2$.

Then $x^4+px^3+qx^2+rx+s$,

and $x^4+2ax^3+(a^2+2b)x^2+2abx+b^2$, are identical;

$$\left.\begin{aligned}\therefore\ p&=2a\\ q&=a^2+2b\\ r&=2ab\\ s&=b^2,\end{aligned}\right\}\qquad \begin{aligned}\therefore\ p^2s&=4a^2b^2,\\ &=r^2.\end{aligned}\qquad \begin{aligned}\text{And }q&=a^2+2b,\\ &=\frac{p^2}{4}+2\sqrt{s}.\end{aligned}$$

Ex. 5. Find the relations subsisting between a, b, c, d, when ax^3+bx^2+cx+d is a complete cube.

Assuming ax^3+bx^2+cx+d to be a complete cube, let it be represented by $(mx+n)^3$.

Then ax^3+bx^2+cx+d,

and $m^3x^3+3m^2nx^2+3mn^2x+n^3$, are identical;

$$\left.\begin{aligned}\therefore\ a&=m^3\\ b&=3m^2n\\ c&=3mn^2\\ d&=n^3,\end{aligned}\right\}\qquad \begin{aligned}&\therefore\ \frac{b^3}{c^3}=\frac{27m^6n^3}{27m^3n^6}=\frac{m^3}{n^3}=\frac{a}{d},\\ &\qquad\text{or } ac^3=db^3.\\ &\text{Also } b^2=9m^4n^2=3m^3c=3ac.\end{aligned}$$

Ex. 6. Find the relations subsisting between a, b, c, d, e, when $ax^4+bx^3+cx^2+dx+e$ is a complete 4th power.

Assuming $ax^4+bx^3+cx^2+dx+e$ to be a complete 4th power, let it be represented by $(mx+n)^4$.

Then $ax^4+bx^3+cx^2+dx+e$,

and $m^4x^4+4m^3nx^3+6m^2n^2x^2+4mn^3x+n^4$, are identical;

$$\left.\begin{aligned}\therefore\ a&=m^4\\ b&=4m^3n\\ c&=6m^2n^2\\ d&=4mn^3\\ e&=n^4,\end{aligned}\right\}\qquad \begin{aligned}&\therefore\ bc=24m^5n^3,\text{ and } ad=4m^5n^3,\\ &\qquad\therefore\ bc=6ad\ldots\ldots\ldots\ldots(1)\\ &\text{Also } cd=24m^3n^5,\text{ and } be=4m^3n^5,\\ &\qquad\therefore\ cd=6be\ldots\ldots\ldots\ldots(2)\\ &\text{And } bd=16m^4n^4=16ae\ldots\ldots(3).\end{aligned}$$

SURDS.

Ex. 19. Simplify $\dfrac{ab}{b-c}\pm\sqrt{\dfrac{a^2b^2}{(b-c)^2}-\dfrac{a^2b}{b-c}}$.

$$\text{Ans.}=\frac{ab}{b-c}\pm\frac{ab}{b-c}\sqrt{1-\frac{b-c}{b}},$$

$$=\frac{ab}{b-c}\left\{1\pm\sqrt{\frac{c}{b}}\right\}=\frac{ab(\sqrt{b}\pm\sqrt{c})}{\sqrt{b}(b-c)},$$

$$=\frac{a\sqrt{b}(\sqrt{b}\pm\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}=\frac{a\sqrt{b}}{\sqrt{b}\mp\sqrt{c}}.$$

Ex. 33. Simplify $\dfrac{\frac{1}{4}(\sqrt{5}\mp 1)}{\frac{1}{2\sqrt{2}}\sqrt{5\pm\sqrt{5}}}$.

$$\text{Fraction} = \frac{\sqrt{5}\mp 1}{\sqrt{2}.\sqrt{5\pm\sqrt{5}}} = \sqrt{\frac{(\sqrt{5}\mp 1)^2}{2(5\pm\sqrt{5})}},$$

$$= \sqrt{\frac{6\mp 2\sqrt{5}}{2(5\pm\sqrt{5})}} = \sqrt{\frac{3\mp\sqrt{5}}{5\pm\sqrt{5}}},$$

$$= \sqrt{\frac{(3\mp\sqrt{5})(5\mp\sqrt{5})}{25-5}} = \sqrt{\frac{20\mp 8\sqrt{5}}{20}},$$

$$= \sqrt{1\mp\frac{2}{5}\sqrt{5}}.$$

Ex. 47. Simplify $\dfrac{\sqrt[3]{x^4}+\sqrt[3]{x^2y^2}-\sqrt[3]{x^3y}}{\sqrt[3]{x^4}+\sqrt[3]{xy^3}-\sqrt[3]{x^3y}-\sqrt[3]{y^4}}$.

$$\text{Ans.} = \frac{x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}-xy^{\frac{1}{3}}}{x^{\frac{4}{3}}+x^{\frac{1}{3}}y-xy^{\frac{1}{3}}-y^{\frac{4}{3}}} = \frac{x^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}})}{x^{\frac{1}{3}}(x+y)-y^{\frac{1}{3}}(x+y)},$$

$$= \frac{x^{\frac{2}{3}}(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})}{(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x+y)},$$

$$= \frac{x^{\frac{2}{3}}(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})}{(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})},$$

$$= \frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}}-y^{\frac{2}{3}}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}-\sqrt[3]{y^2}}.$$

Ex. 49. Simplify $\sqrt{a^2+\sqrt[3]{a^4b^2}}+\sqrt{b^2+\sqrt[3]{a^2b^4}}$.

$$\text{Ans.} = \sqrt{a^2+a^{\frac{4}{3}}b^{\frac{2}{3}}}+\sqrt{b^2+a^{\frac{2}{3}}b^{\frac{4}{3}}},$$

$$= \sqrt{a^{\frac{4}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})}+\sqrt{b^{\frac{4}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})},$$

$$= \sqrt{a^{\frac{4}{3}}+b^{\frac{4}{3}}.(a^{\frac{2}{3}}+b^{\frac{2}{3}})},$$

$$= (a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}.$$

Ex. 72. Find the square root of

$$\sqrt[x]{a^{-\frac{2}{y}}}+\sqrt[r]{a^{\frac{1}{y}}b}+2\sqrt[2]{\sqrt[r]{b}}.\sqrt[2]{\sqrt[x]{\sqrt[y]{\sqrt[r]{a^{x-2r}}}}}.$$

The proposed quantity properly arranged with fractional indices is

$$a^{-\frac{2}{xy}}+2a^{\frac{x-2r}{2xyr}}b^{\frac{1}{2r}}+a^{\frac{1}{ry}}b^{\frac{1}{r}},$$

$$=(a^{-\frac{1}{xy}})^2+2a^{-\frac{1}{xy}}.a^{\frac{1}{2ry}}b^{\frac{1}{2r}}+(a^{\frac{1}{2ry}}.b^{\frac{1}{2r}})^2,$$

$$=(a^{-\frac{1}{xy}}+a^{\frac{1}{2ry}}b^{\frac{1}{2r}})^2,$$

∴ square root required

$$=a^{-\frac{1}{xy}}+a^{\frac{1}{2ry}}b^{\frac{1}{2r}}=\sqrt[x]{a^{-\frac{1}{y}}}+\sqrt[2r]{a^{\frac{1}{y}}b}.$$

Ex. 85. Reduce $\dfrac{\sqrt[3]{5\cdot12}+\sqrt[3]{0\cdot03375}}{\sqrt[3]{80}-\sqrt[3]{0\cdot01}}$.

Multiplying Numerator and Denominator by $\sqrt[3]{100}$, the fraction becomes

$$\frac{\sqrt[3]{512}+\sqrt[3]{3\cdot375}}{\sqrt[3]{8000}-\sqrt[3]{1}}=\frac{8+1\cdot5}{20-1}=\frac{9\cdot5}{19}=0\cdot5,\text{ or }\frac{1}{2}.$$

Ex. 87. Prove that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{2-\sqrt{2-\sqrt{3}}}$ is equal to $\sqrt{2}$.

$$\because \sqrt{2+\sqrt{3}}=\sqrt{\frac{4+2\sqrt{3}}{2}}=\frac{\sqrt{3}+1}{\sqrt{2}},$$

$$\text{and } \sqrt{2-\sqrt{3}}=\sqrt{\frac{4-2\sqrt{3}}{2}}=\frac{\sqrt{3}-1}{\sqrt{2}},$$

$$\therefore \text{Sum of fractions}=\frac{2+\sqrt{3}}{\sqrt{2}+\frac{\sqrt{3}+1}{\sqrt{2}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\frac{\sqrt{3}-1}{\sqrt{2}}},$$

$$=\sqrt{2}\times\left\{\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2-\sqrt{3}}{3-\sqrt{3}}\right\}=\sqrt{2}\times\left\{\frac{6-3+6-3}{9-3}\right\},$$

$$=\sqrt{2}\times\frac{6}{6}=\sqrt{2}.$$

Ex. 92. Simplify $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$.

Fraction $=\dfrac{3+\sqrt{5}+5-\sqrt{5}-2\sqrt{3+\sqrt{5}}.\sqrt{5-\sqrt{5}}}{3+\sqrt{5}-(5-\sqrt{5})}$, multiplying Num^r. and Denom^r. by the Num^r.

$$=\frac{8-2\sqrt{10+2\sqrt{5}}}{2\sqrt{5}-2},$$

$$=\frac{4-\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1},$$

$$=\frac{4(\sqrt{5}+1)-(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}}{4},$$

$$=\sqrt{5}+1-\frac{1}{4}\sqrt{6+2\sqrt{5}}.\sqrt{10+2\sqrt{5}},$$

$$=\sqrt{5}+1-\frac{1}{4}\sqrt{80+32\sqrt{5}},$$

$$=\sqrt{5}+1-\sqrt{5+2\sqrt{5}}.$$

Ex. 93. Simplify $\sqrt{\dfrac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}$.

Fraction $=\dfrac{2-\sqrt{2+\sqrt{2}}}{\sqrt{4-(2+\sqrt{2})}}$, multip^s. num^r. & denom^r. by the num^r.

$$=\frac{2-\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}},$$

$$=\frac{2\sqrt{2-\sqrt{2}}-\sqrt{2}}{2-\sqrt{2}},$$

$$=\frac{2\sqrt{2+\sqrt{2}}.\sqrt{2+\sqrt{2}}.\sqrt{2-\sqrt{2}}-(2+\sqrt{2})\sqrt{2}}{4-2},$$

$$=\sqrt{2+\sqrt{2}}.\sqrt{2}-\sqrt{2}-1.$$

Ex. 97. Simplify $\dfrac{\sqrt{1+a^2}-\sqrt{1-a^2}}{a^2+2-2\sqrt{1-a^4}}$.

Fraction $=\dfrac{(\sqrt{1+a^2}-\sqrt{1-a^2})(a^2+2+2\sqrt{1-a^4})}{a^4+4a^2+4-(4-4a^4)}$,

$$=\frac{(\sqrt{1+a^2}-\sqrt{1-a^2})\{a^2+(\sqrt{1+a^2}+\sqrt{1-a^2})^2\}}{5a^4+4a^2},$$

$$=\frac{a^2(\sqrt{1+a^2}-\sqrt{1-a^2})+2a^2(\sqrt{1+a^2}+\sqrt{1-a^2})}{5a^4+4a^2},$$

$$=\frac{3\sqrt{1+a^2}+\sqrt{1-a^2}}{5a^2+4}.$$

Ex. 100. Prove that $5^{\frac{3}{4}}.\left\{\frac{\sqrt{5}\pm1}{2}\right\}^{\frac{1}{2}}$ is equivalent to $\frac{5}{2}\sqrt{2\pm\frac{2}{5}\sqrt{5}}$; and $5^{\frac{3}{4}}.\left\{\frac{\sqrt{5}\pm1}{2}\right\}^{\frac{5}{2}}$ to $\frac{5}{2}\sqrt{10\pm\frac{22}{5}\sqrt{5}}$.

$$(1)\quad 5^{\frac{3}{4}}.\sqrt{\frac{\sqrt{5}\pm1}{2}}=\frac{5}{\sqrt[4]{5}}\sqrt{\frac{\sqrt{5}\pm1}{2}}=\frac{5}{\sqrt{2}}\sqrt{1\pm\frac{1}{\sqrt{5}}},$$

$$=\frac{5}{2}\sqrt{2\pm\frac{2}{\sqrt{5}}}=\frac{5}{2}\sqrt{2\pm\frac{2}{5}\sqrt{5}}.$$

$$(2)\quad 5^{\frac{3}{4}}.\left(\frac{\sqrt{5}\pm1}{2}\right)^{\frac{5}{2}}=5^{\frac{3}{4}}.\left(\frac{\sqrt{5}\pm1}{2}\right)^{2}.\sqrt{\frac{\sqrt{5}\pm1}{2}}=\frac{3\pm\sqrt{5}}{2}.5^{\frac{3}{4}}\sqrt{\frac{\sqrt{5}\pm1}{2}},$$

$$=\frac{5}{2}\sqrt{\frac{(3\pm\sqrt{5})^2}{4}.(2\pm\frac{2}{5}\sqrt{5})},\ \text{from (1)},$$

$$=\frac{5}{2}\sqrt{(7\pm3\sqrt{5})(1\pm\frac{1}{5}\sqrt{5})},$$

$$=\frac{5}{2}\sqrt{10\pm\frac{22}{5}\sqrt{5}}.$$

Ex. 101. Find the value of $\left(\frac{1}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2$, when $x=\sqrt{\frac{n-1}{n+1}}$.

$$\frac{x}{x-1}=\frac{\sqrt{n-1}}{\sqrt{n-1}-\sqrt{n+1}},\ \text{and}\ \frac{x}{x+1}=\frac{\sqrt{n-1}}{\sqrt{n-1}+\sqrt{n+1}},$$

$$\therefore\ \text{Value required}=\frac{n-1}{2n-2\sqrt{n^2-1}}+\frac{n-1}{2n+2\sqrt{n^2-1}},$$

$$=\frac{n-1}{2}\left\{\frac{1}{n-\sqrt{n^2-1}}+\frac{1}{n+\sqrt{n^2-1}}\right\},$$

$$=\frac{n-1}{2}\times\frac{2n}{1}=n(n-1).$$

Ex. 102. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when

$$x=\frac{12ab}{a+b+\sqrt{\{(a+b)^2+12ab\}}}.$$

$$\text{Sum of fractions} = \frac{2x^2-8ab}{x^2-2(a+b)x+4ab}.$$

$$\text{If } x=\frac{12ab}{a+b+\sqrt{(a+b)^2+12ab}}=\frac{12ab\{a+b-\sqrt{(a+b)^2+12ab}\}}{(a+b)^2-\{(a+b)^2+12ab\}},$$

$$=\sqrt{(a+b)^2+12ab}-(a+b),$$

$$\text{then } 2x^2-8ab=2(a+b)^2+16ab+2(a+b)^2-4(a+b)\sqrt{(a+b)^2+12ab},$$

$$=4(a+b)^2+16ab-4(a+b)\sqrt{(a+b)^2+12ab}.$$

$$\text{And } \{x-(a+b)\}^2=x^2-2(a+b)x+(a+b)^2,$$

$$=(a+b)^2+12ab+4(a+b)^2-4(a+b)\sqrt{(a+b)^2+12ab};$$

$$\therefore\ x^2-2(a+b)x+4ab=4(a+b)^2+16ab-4(a+b)\sqrt{(a+b)^2+12ab}.$$

Hence Ans. $=1$,

Ex. 103. Find the value of $\dfrac{x^{\frac{1}{2}}+x}{1+x^{\frac{3}{2}}}$, when $x=\frac{1}{2}(a^2+a\sqrt{a^2+4})$.

$$\frac{x^{\frac{1}{2}}+x}{1+x^{\frac{3}{2}}}=x^{\frac{1}{2}}.\frac{1+x^{\frac{1}{2}}}{1+x^{\frac{3}{2}}}=\frac{x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}+x}=\frac{1}{x^{\frac{1}{2}}+\frac{1}{x^{\frac{1}{2}}}-1}$$

$$=\frac{1}{\sqrt{x+\frac{1}{2}+2}-1}\ \ldots\ldots\ldots\ldots(1).$$

But $x-\dfrac{a^2}{2}=\dfrac{a}{2}\sqrt{a^2+4}$, and squaring these equals,

$$x^2-a^2x=a^2,\quad\text{and } \frac{a^2}{x}=x-a^2,\quad\text{or } a^2+\frac{a^2}{x}=x.$$

Now multiplying Numerator and Denominator of (1) by a, it becomes

$$\frac{a}{\sqrt{a^2x+\frac{a^2}{x}+2a^2}-a}=\frac{a}{\sqrt{x^2+a^2+\frac{a^2}{x}}-a}$$

$$=\frac{a}{\sqrt{x^2+x}-a}=\frac{2a}{\sqrt{4x^2+4x}-2a}=\frac{2a}{\sqrt{(2x+1)^2-1}-2a}$$

$$=\frac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1}-2a}.$$

Ex. 104. Simplify $\dfrac{n^3-3n+(n^2-1)\sqrt{n^2-4}-2}{n^3-3n+(n^2-1)\sqrt{n^2-4}+2}$.

$$\text{Fraction}=\frac{(n^2+2n+1)n-(n^2+2n+1)2+(n^2-1)\sqrt{n^2-4}}{(n^2-2n+1)n+(n^2-2n+1)2+(n^2-1)\sqrt{n^2-4}},$$

$$=\frac{(n+1)^2(n-2)+(n^2-1)\sqrt{n^2-4}}{(n-1)^2(n+2)+(n^2-1)\sqrt{n^2-4}},$$

$$=\frac{(n+1)\sqrt{n-2}.\{(n+1)\sqrt{n-2}+(n-1)\sqrt{n+2}\}}{(n-1)\sqrt{n+2}.\{(n-1)\sqrt{n+2}+(n+1)\sqrt{n-2}\}},$$

$$=\frac{n+1}{n-1}\sqrt{\frac{n-2}{n+2}}.$$

Ex. 105. Simplify $\left\{\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}\right\}^{\frac{1}{2}}$.

$$\text{Ans.}=\left\{1-\frac{2}{2x^2+3+x\sqrt{4x^2+3}}\right\}^{\frac{1}{2}},$$

$$=\left\{1-\frac{4x^2+6-2x\sqrt{4x^2+3}}{4x^4+12x^2+9-(4x^4+3x^2)}\right\}^{\frac{1}{2}},$$

$$=\left\{1-\frac{4x^2+6-2x\sqrt{4x^2+3}}{9x^2+9}\right\}^{\frac{1}{2}},$$

$$=\left\{\frac{5x^2+3+2x\sqrt{4x^2+3}}{9(x^2+1)}\right\}^{\frac{1}{2}}=\frac{x+\sqrt{4x^2+3}}{3\sqrt{x^2+1}}.$$

Ex. 106. Simplify $\dfrac{xy-\sqrt{\dfrac{1-x^2}{1+x^2}\cdot\dfrac{1-y^2}{1+y^2}}}{1+xy\sqrt{\dfrac{1-x^2}{1+x^2}\cdot\dfrac{1-y^2}{1+y^2}}}$.

$$\text{Fraction}=\frac{xy\sqrt{(1+x^2)(1+y^2)}-\sqrt{(1-x^2)(1-y^2)}}{\sqrt{(1+x^2)(1+y^2)}+xy\sqrt{(1-x^2)(1-y^2)}},$$

$$=\frac{xy(1+x^2)(1+y^2)+xy(1-x^2)(1-y^2)-x^2y^2\sqrt{(1-x^4)(1-y^4)}-\sqrt{(1-x^4)(1-y^4)}}{\{\sqrt{(1+x^2)(1+y^2)}\}^2-\{xy\sqrt{(1-x^2)(1-y^2)}\}^2},$$

$$=\frac{2xy(1+x^2y^2)-(1+x^2y^2)\sqrt{(1-x^4)(1-y^4)}}{1+x^2+y^2+x^2y^2-x^2y^2(1-x^2-y^2+x^2y^2)},$$

$$=\frac{(1+x^2y^2)\{2xy-\sqrt{(1-x^4)(1-y^4)}\}}{1+x^2+y^2-x^2y^2+x^2y^2(1+x^2+y^2-x^2y^2)},$$

$$=\frac{(1+x^2y^2)\{2xy-\sqrt{(1-x^4)(1-y^4)}\}}{(1+x^2y^2)(1+x^2+y^2-x^2y^2)}=\frac{2xy-\sqrt{(1-x^4)(1-y^4)}}{1+x^2+y^2-x^2y^2}.$$

Ex. 107. Simplify $\dfrac{1-ab+\sqrt{1+a^2}-a\sqrt{1+b^2}}{1-ab+\sqrt{1+b^2}-b\sqrt{1+a^2}}$.

Numerator

$$=1-a+\sqrt{1+a^2}+a(1-b-\sqrt{1+b^2}),$$

$$=1-a+\sqrt{1+a^2}-\frac{1}{2}(1-a+\sqrt{1+a^2})(1-a-\sqrt{1+a^2})(1-b-\sqrt{1+b^2}),$$

$$=(1-a+\sqrt{1+a^2})\{1-\frac{1}{2}(1-a-\sqrt{1+a^2})(1-b-\sqrt{1+b^2})\};$$

and Denominator = Numerator with a for b, and b for a,

$$=(1-b+\sqrt{1+b^2})\{1-\frac{1}{2}(1-b-\sqrt{1+b^2})(1-a-\sqrt{1+a^2})\};$$

$$\therefore \text{ Fraction}=\frac{1-a+\sqrt{1+a^2}}{1-b+\sqrt{1+b^2}}.$$

Ex. 108. Find the value of $\sqrt{\dfrac{n^2-1}{n^2+1}}$, when $n^2=\dfrac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$.

$$\frac{n^2-1}{n^2+1}=\frac{2n^2-2}{2n^2+2}=\frac{\sqrt{3}-1+\sqrt{2\sqrt{3}}}{\sqrt{3}+3+\sqrt{2\sqrt{3}}},$$

$$=1-\frac{4}{\sqrt{3}+3+\sqrt{2\sqrt{3}}},$$

$$=1-\frac{4(\sqrt{3}+3-\sqrt{2\sqrt{3}}}{12+6\sqrt{3}-2\sqrt{3}},$$

$$=1-\frac{\sqrt{3}+3-\sqrt{2\sqrt{3}}}{3+\sqrt{3}},$$

$$=\frac{\sqrt{2\sqrt{3}}}{3+\sqrt{3}}=\frac{\sqrt{2\sqrt{3}}}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}+1},$$

$$=\frac{\sqrt{2\sqrt{3}}}{\sqrt{3}}\cdot\frac{\sqrt{3}-1}{3-1},$$

$$=\frac{\sqrt{2\sqrt{3}}}{2\sqrt{3}}(\sqrt{3}-1),$$

$$=\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}},$$

$$\therefore \text{ Value required}=\sqrt{\frac{\sqrt{3}-1}{2\sqrt{3}}}.$$

Ex. 109. Find the value of $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}$, when $x=\frac{1}{a}\sqrt{\frac{2a}{b}-1}$.

$$(1-ax)\sqrt{1+bx}=\left(1-\sqrt{\frac{2a}{b}-1}\right)\cdot\sqrt{1+\frac{b}{a}\sqrt{\frac{2a}{b}-1}},$$

$$=\sqrt{\left(\frac{2a}{b}-2\sqrt{\frac{2a}{b}-1}\right)\left(1+\frac{b}{a}\sqrt{\frac{2a}{b}-1}\right)},$$

$$=\sqrt{\frac{2a}{b}-\frac{2b}{a}\left(\frac{2a}{b}-1\right)},$$

$$=\sqrt{\frac{2a}{b}-4+\frac{2b}{a}},$$

and $(1+ax)\sqrt{1-bx}$ is the same as $(1-ax)\sqrt{1+bx}$ with $-a$ for a, and $-b$ for b, that is,

$$=\sqrt{\frac{2a}{b}-4+\frac{2b}{a}},$$

$$\therefore \text{ value required}=1.$$

Ex. 111. Find the value of $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x=\frac{2ab}{b^2+1}$.

$$\text{Fraction}=\frac{(\sqrt{a+x}+\sqrt{a-x})^2}{a+x-(a-x)}=\frac{2a+2\sqrt{a^2-x^2}}{2x},$$

$$=\frac{a+\sqrt{a^2-x^2}}{x}=\frac{a}{x}+\sqrt{\frac{a^2}{x^2}-1},$$

$$\therefore \text{ Ans.} = \frac{b^2+1}{2b} + \sqrt{\frac{b^4+2b^2+1}{4b^2} - 1},$$

$$= \frac{b^2+1}{2b} + \frac{b^2-1}{2b} = \frac{2b^2}{2b} = b.$$

Ex. 112. Find the value of $2(uv - \sqrt{1-u^2}.\sqrt{1-v^2})$, when $2u = x + x^{-1}$, and $2v = y + y^{-1}$.

$$\sqrt{1-u^2} = \sqrt{1 - \frac{1}{4}(x^2 + x^{-2} + 2)},$$

$$= \sqrt{\frac{1}{2} - \frac{x^2}{4} - \frac{x^{-2}}{4}} = \sqrt{-\frac{1}{4}(x^2 - 2 + x^{-2})},$$

$$= \frac{\sqrt{-1}}{2}(x - x^{-1}).$$

Similarly, $\sqrt{1-v^2} = \frac{\sqrt{-1}}{2}(y - y^{-1})$.

$\therefore$ Value required

$$= \frac{1}{2}(x + x^{-1})(y + y^{-1}) + \frac{1}{2}(x - x^{-1})(y - y^{-1}),$$

$$= \frac{1}{2}\{xy + (xy)^{-1}\} + \frac{1}{2}(xy^{-1} + x^{-1}y) + \frac{1}{2}\{xy + (xy)^{-1}\} - \frac{1}{2}(xy^{-1} + x^{-1}y),$$

$$= xy + (xy)^{-1}.$$

Ex 113. Find the value of $\frac{2a\sqrt{1+x^2}}{x + \sqrt{1+x^2}}$, when $x = \frac{1}{2}\left\{\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right\}$.

Proposed fraction

$$= \frac{2a\sqrt{1+x^2}(x - \sqrt{1+x^2})}{-1}, \text{ and } x^2 = \frac{1}{4}\left(\frac{a}{b} + \frac{b}{a} - 2\right),$$

$$\therefore \sqrt{1+x^2} = \sqrt{\frac{1}{4}\left(\frac{a}{b} + \frac{b}{a} + 2\right)} = \frac{1}{2}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right),$$

$\therefore$ Value required $= 2a(1+x^2) - 2ax\sqrt{1+x^2}$,

$$= 2a.\frac{1}{4}\left(\frac{a}{b} + \frac{b}{a} + 2\right) - 2a.\frac{1}{4}\left(\frac{a}{b} - \frac{b}{a}\right),$$

$$= 2a\left(\frac{1}{2}.\frac{b}{a} + \frac{1}{2}\right) = a + b.$$

Ex. 115. If $x=\left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$, find the value of $\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x})$.

$$\sqrt[p]{x}=\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}, \text{ and } \sqrt[q]{x}=\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}};$$

$$\therefore \text{ value required }=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\left\{\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}+\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}}\right\},$$

$$=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}}\cdot\left\{1+\left(\frac{a+b}{a-b}\right)^2\right\},$$

$$=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}}\cdot\frac{2(a^2+b^2)}{(a-b)^2},$$

$$=\frac{a+b}{a-b}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}}=\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}+1},$$

$$=\left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}.$$

EQUATIONS.

[So many Solutions of Equations are given in the *Appendix* to the Algebra, that a few only will be inserted here.]

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

Ex. 72. If $\frac{1}{a}\sqrt{a+x}+\frac{1}{x}\sqrt{a+x}=\frac{1}{b}\sqrt{x}$, find x.

$$\left(\frac{1}{a}+\frac{1}{x}\right)\sqrt{a+x}=\frac{1}{b}\sqrt{x},$$

$$(a+x)\sqrt{a+x}=\frac{a}{b}x\sqrt{x},$$

$$(a+x)^{\frac{3}{2}}=\frac{a}{b}x^{\frac{3}{2}},$$

$$b^{\frac{2}{3}}(a+x)=a^{\frac{2}{3}}x,$$

$$\therefore x=\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}.$$

Ex. 82. If $\dfrac{ax-b^2}{\sqrt{ax}+b}=\dfrac{\sqrt{ax}-b}{n}-c$, find x.

$$\sqrt{ax}-b=\frac{\sqrt{ax}-b}{n}-c,$$

$$(\sqrt{ax}-b)\left(1-\frac{1}{n}\right)=-c,$$

$$\sqrt{ax}-b=-\frac{nc}{n-1},$$

$$ax=\left(b-\frac{nc}{n-1}\right)^2,$$

$$\therefore\ x=\frac{1}{a}\left(b-\frac{nc}{n-1}\right)^2.$$

Ex. 85. If $\sqrt[m]{a+x}=\sqrt[2m]{x^2+8ax+b^2}$, find x.

Raising both sides to the $2m^{\text{th}}$ power,

$$(a+x)^2=x^2+8ax+b^2,$$

$$a^2+2ax=8ax+b^2,$$

$$6ax=a^2-b^2,$$

$$\therefore\ x=\frac{a^2-b^2}{6a}.$$

Ex. 87. This is similar to Ex. 82.

Ex. 90. If $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$, find x.

$$\sqrt{\frac{1+bx}{1-bx}}=\frac{1+ax}{1-ax},$$

$$\frac{1+bx}{1-bx}=\frac{1+2ax+a^2x^2}{1-2ax+a^2x^2},$$

$$\frac{bx}{1}=\frac{2ax}{1+a^2x^2},\ \text{(Art. 195)}.$$

$$1+a^2x^2=\frac{2a}{b},$$

$$a^2x^2=\frac{2a}{b}-1,$$

$$\therefore\ x=\frac{1}{a}\sqrt{\frac{2a}{b}-1}.$$

Ex. 91. If $\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{62}{63}\cdot\dfrac{1+x}{1-x}$, find x.

$$\frac{62}{63}=\frac{(1-x)(1+x+x^2)}{(1+x)(1-x+x^2)}=\frac{1-x^3}{1+x^3},$$

$$\frac{2x^3}{2}=\frac{63-62}{63+62},$$

$$\text{or } x^3=\frac{1}{125}; \quad \therefore\ x=\sqrt[3]{\frac{1}{125}}=\frac{1}{5}.$$

Ex. 92. If $\left(\dfrac{a+x}{a-x}\right)^2=1+\dfrac{cx}{ab}$, find x.

$$1+\frac{cx}{ab}=\left(1+\frac{2x}{a-x}\right)^2,$$

$$=1+\frac{4x}{a-x}+\frac{4x^2}{(a-x)^2};$$

$$\therefore\ \frac{c}{ab}=\frac{4}{a-x}+\frac{4x}{(a-x)^2},$$

$$=\frac{4a}{(a-x)^2},$$

$$\frac{c}{b}=\frac{4a^2}{(a-x)^2},$$

$$\therefore\ a-x=\pm 2a\sqrt{\frac{b}{c}},$$

$$\therefore\ x=a\left\{1\mp 2\sqrt{\frac{b}{c}}\right\}.$$

Ex. 96. If $a+b=\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, find x.

$$(a+b)x+(a+b)\sqrt{1+x^2}=2a\sqrt{1+x^2},$$

$$(a+b)x=(a-b)\sqrt{1+x^2},$$

$$\frac{\sqrt{1+x^2}}{x}=\frac{a+b}{a-b},$$

$$\frac{1}{x^2}+1=\left(\frac{a+b}{a-b}\right)^2,$$

$$\frac{1}{x^2}=\left(\frac{a+b}{a-b}\right)^2-1=\frac{4ab}{(a-b)^2},$$

$$\therefore\ x=\frac{a-b}{2\sqrt{ab}}=\frac{1}{2}\left\{\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right\}.$$

Ex. [illegible] If [illegible] $= \frac{\sqrt{x^2-1}-1}{x^2-2}$, find x.

For [illegible] write y, and [illegible] y^2-1 for x^2-2,

then [illegible] $= \frac{y-1}{y^2-1}$,

$= \frac{1}{y+1}$.

[illegible]

[illegible]

$\therefore y =$ [illegible],

or $\sqrt{x^2-1} =$ [illegible],

$x^2-1 =$ [illegible],

$\therefore x = \sqrt{\text{[illegible]}}$.

Ex. [illegible] If $\sqrt{1-a}\cdot\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \sqrt{1-a}\cdot\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} = 2\sqrt[4]{1-a^2}$, find x.

For $\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}$ write y, and $\therefore \frac{1}{y}$ for $\left(\frac{1-x}{1+x}\right)^{\frac{1}{4}}$,

$$\text{then } \sqrt{1-a}\cdot y - \sqrt{1-a}\cdot\frac{1}{y} = 2\sqrt[4]{1-a^2},$$

$$\therefore \sqrt{1-a}\cdot y^2 - 2\sqrt[4]{1-a^2}\cdot y + \sqrt{1-a} = 0,$$

extracting the square root $\sqrt[4]{1-a}\cdot y - \sqrt[4]{1-a} = 0$;

$$\therefore y = \sqrt[4]{\frac{1-a}{1+a}},$$

$$\therefore \frac{1+x}{1-x} = y^4 = \frac{1-a}{1+a},$$

$$\therefore x = -a.$$

Ex. 127. If $\frac{a+c(a+x)}{a+c(a-x)} + \frac{a+x}{x} = \frac{a}{a-2cx}$, find x.

$$\frac{a+x}{x} = \frac{a}{a-2cx} - \frac{a+ac+cx}{a+ac-cx},$$

$$= \frac{a^2+a^2c-acx-(a^2+a^2c-acx-2ac^2x-2c^2x^2)}{(a-2cx)(a+ac-cx)},$$

$$= \frac{2c^2x(a+x)}{(a-2cx)(a+ac-cx)},$$

$$\therefore\ \frac{1}{x}=\frac{2c^2x}{(a-2cx)(a+ac-cx)},\ \text{and } a+x=0, \text{ or } x=-a.$$

$$a^2+a^2c-acx-2acx-2ac^2x+2c^2x^2=2c^2x^2,$$

$$(3c+2c^2)x=a+ac,$$

$$\therefore\ x=\frac{a}{c}\cdot\frac{1+c}{3+2c}.$$

Ex. 128. If $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$, find x.

$$\left(1-\frac{a-b}{x-b}\right)^3=1-\frac{3(a-b)}{x-2b+a},$$

$$=1-\frac{3\cdot\dfrac{a-b}{x-b}}{1+\dfrac{a-b}{x-b}};\ \text{for } \frac{a-b}{x-b} \text{ put } y,$$

$$(1-y)^3=1-\frac{3y}{1+y},$$

$$1-3y+3y^2-y^3=1-\frac{3y}{1+y},$$

$$-3+3y-y^2=-\frac{3}{1+y},$$

$$\left.\begin{array}{r}-3+3y-y^2\\ -3y+3y^2-y^3\end{array}\right\}=-3,$$

$$2y^2=y^3,\quad \therefore\ y=2,\quad \frac{a-b}{x-b}=2,\quad \therefore\ x=\frac{1}{2}(a+b).$$

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

Ex. 16. If $2{\cdot}4x+0{\cdot}32y-\dfrac{0{\cdot}36x-0{\cdot}05}{0{\cdot}5}=0{\cdot}8x+\dfrac{2{\cdot}6+0{\cdot}005y}{0{\cdot}25}$ ...(1),

and $\dfrac{0{\cdot}04y+0{\cdot}1}{0{\cdot}3}=\dfrac{0{\cdot}07x-0{\cdot}1}{0{\cdot}6}$ ....(2),

find x and y.

From (1), multiplying numerator and denominator of the first fraction by 2, and of the second by 4, we have

$$2{\cdot}4x+0{\cdot}32y-0{\cdot}72x+0{\cdot}1=0{\cdot}8x+10{\cdot}4+0{\cdot}02y,$$

$$0{\cdot}88x+0{\cdot}3y=10{\cdot}3,$$

$$\text{or } 88x+30y=1030 \text{ ....(3).}$$

Ex. 98. If $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{\sqrt{x^2-1}-1}{x^2-2}$, find x.

For $\sqrt{x^2-1}$ write y, and $\therefore\ y^2-1$ for x^2-2,

$$\text{then } \frac{1+y}{1+2ay}=\frac{y-1}{y^2-1},$$

$$=\frac{1}{y+1},$$

$$\therefore\ y^2+2y+1=1+2ay,$$

$$y+2=2a,$$

$$\therefore\ y=2(a-1),$$

$$\text{or } \sqrt{x^2-1}=2(a-1),$$

$$x^2-1=4(a-1)^2,$$

$$\therefore\ x=\sqrt{1+4(a-1)^2}.$$

Ex. 106. If $\sqrt{1+a}.\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}+\sqrt{1-a}.\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{2}}=2\sqrt[4]{1-a^2}$, find x.

For $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$ write y, and $\therefore\ \dfrac{1}{y}$ for $\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{2}}$,

$$\text{then } \sqrt{1+a}.y+\sqrt{1-a}.\frac{1}{y}=2\sqrt[4]{1-a^2},$$

$$\therefore\ \sqrt{1+a}.y^2-2\sqrt[4]{1-a^2}.y+\sqrt{1-a}=0,$$

extracting the square root $\sqrt[4]{1+a}.y-\sqrt[4]{1-a}=0$;

$$\therefore\ y=\sqrt[4]{\frac{1-a}{1+a}},$$

$$\therefore\ \frac{1+x}{1-x}=y^4=\frac{1-a}{1+a},$$

$$\therefore\ x=-a.$$

Ex. 127. If $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$, find x.

$$\frac{a+x}{x}=\frac{a}{a-2cx}-\frac{a+ac+cx}{a+ac-cx},$$

$$=\frac{a^2+a^2c-acx-(a^2+a^2c-acx-2ac^2x-2c^2x^2)}{(a-2cx)(a+ac-cx)},$$

$$=\frac{2c^2x(a+x)}{(a-2cx)(a+ac-cx)},$$

$$\therefore \frac{1}{x}=\frac{2c^2x}{(a-2cx)(a+ac-cx)}, \text{ and } a+x=0, \text{ or } x=-a.$$

$$a^2+a^2c-acx-2acx-2ac^2x+2c^2x^2=2c^2x^2,$$

$$(3c+2c^2)x=a+ac,$$

$$\therefore x=\frac{a}{c}\cdot\frac{1+c}{3+2c}.$$

Ex. 128. If $\left(\frac{x-a}{x-b}\right)^3=\frac{x-2a+b}{x-2b+a}$, find x.

$$\left(1-\frac{a-b}{x-b}\right)^3=1-\frac{3(a-b)}{x-2b+a},$$

$$=1-\frac{3.\frac{a-b}{x-b}}{1+\frac{a-b}{x-b}}; \text{ for } \frac{a-b}{x-b} \text{ put } y,$$

$$(1-y)^3=1-\frac{3y}{1+y},$$

$$1-3y+3y^2-y^3=1-\frac{3y}{1+y},$$

$$-3+3y-y^2=-\frac{3}{1+y},$$

$$\left.\begin{array}{r}-3+3y-y^2\\ -3y+3y^2-y^3\end{array}\right\}=-3,$$

$$2y^2=y^3, \quad \therefore y=2, \quad \frac{a-b}{x-b}=2, \quad \therefore x=\frac{1}{2}(a+b).$$

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

Ex. 16. If
$$\left.\begin{array}{l}2{\cdot}4x+0{\cdot}32y-\dfrac{0{\cdot}36x-0{\cdot}05}{0{\cdot}5}=0{\cdot}8x+\dfrac{2{\cdot}6+0{\cdot}005y}{0{\cdot}25}\ldots(1),\\ \text{and } \dfrac{0{\cdot}04y+0{\cdot}1}{0{\cdot}3}=\dfrac{0{\cdot}07x-0{\cdot}1}{0{\cdot}6}\ldots\ldots\ldots\ldots(2),\end{array}\right\}$$
find x and y.

From (1), multiplying numerator and denominator of the first fraction by 2, and of the second by 4, we have

$$2{\cdot}4x+0{\cdot}32y-0{\cdot}72x+0{\cdot}1=0{\cdot}8x+10{\cdot}4+0{\cdot}02y,$$

$$0{\cdot}88x+0{\cdot}3y=10{\cdot}3,$$

$$\text{or } 88x+30y=1030 \ldots\ldots\ldots\ldots(3).$$

$$\text{From (2)} \quad 0{\cdot}08y+0{\cdot}2=0{\cdot}07x-0{\cdot}1,$$
$$0{\cdot}07x-0{\cdot}08y=0{\cdot}3,$$
$$\text{or } 7x-8y=30 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$
$$\therefore \text{ from (3)} \quad 704x+240y=8240,$$
$$\text{and from (4)} \quad 210x-240y=\ 900,$$
$$\therefore\ 914x=9140,$$
$$\therefore\ x=10.$$
$$\text{And } 8y=7x-30=70-30=40,$$
$$\therefore\ y=5.$$

Ex. 22. If $\left.\begin{array}{l} \dfrac{5\sqrt{x+y}}{x}+\dfrac{5\sqrt{x+y}}{y}=10\tfrac{2}{3}, \\ \text{and } \dfrac{3\sqrt{x-y}}{y}-\dfrac{3\sqrt{x-y}}{x}=\dfrac{4}{5}, \end{array}\right\}$ find x and y.

$$\left.\begin{array}{ll} \text{From 1st} & 5(x+y)\sqrt{x+y}=\dfrac{32}{3}xy, \\ \ldots\ \text{2nd} & 3(x+y)\sqrt{x-y}=\dfrac{4}{5}xy, \end{array}\right\}$$

$$\therefore\ \frac{5}{3}\left(\frac{x+y}{x-y}\right)^{\frac{3}{2}}=\frac{32}{3}\times\frac{5}{4},$$

$$\left(\frac{x+y}{x-y}\right)^{\frac{3}{2}}=\frac{32}{3}\times\frac{5}{4}\times\frac{3}{5}=8,$$

$$\frac{x+y}{x-y}=8^{\frac{2}{3}}=4,$$

$$\therefore\ \frac{x}{y}=\frac{5}{3}, \text{ or } y=\frac{3}{5}x.$$

$$\therefore \text{ substituting for } y, \quad 3\left(x-\frac{3}{5}x\right)^{\frac{3}{2}}=\frac{4}{5}x\cdot\frac{3}{5}x,$$

$$\left(\frac{2x}{5}\right)^{\frac{3}{2}}=\frac{4x^2}{25}=\left(\frac{2x}{5}\right)^2,$$

$$\therefore\ \frac{2x}{5}=1, \text{ or } x=\frac{5}{2}=2\tfrac{1}{2}.$$

$$\text{Also } y=\frac{3x}{5}=\frac{3}{5}\times\frac{5}{2}=\frac{3}{2}=1\tfrac{1}{2}.$$

Ex. 23. $\left.\begin{array}{l}\text{If } axy = c(bx+ay)\ldots\ldots(1)\\ \text{and } bxy = c(ax-by)\ldots\ldots(2)\end{array}\right\}$ find x and y.

Divide both by cxy, then

$$\left.\begin{array}{l}\dfrac{a}{c}=\dfrac{b}{y}+\dfrac{a}{x},\\ \text{and } \dfrac{b}{c}=\dfrac{a}{y}-\dfrac{b}{x},\end{array}\right\}$$

$$\left.\begin{array}{l}\therefore\ \dfrac{a^2}{c}=\dfrac{ab}{y}+\dfrac{a^2}{x},\\ \text{and } \dfrac{b^2}{c}=\dfrac{ab}{y}-\dfrac{b^2}{x}.\end{array}\right\}$$

$$\therefore\ \frac{a^2-b^2}{c}=\frac{a^2+b^2}{x};\quad \therefore\ x=\frac{a^2+b^2}{a^2-b^2}.c.$$

$$\text{Similarly } y=\frac{a^2+b^2}{2ab}.c.$$

Ex. 27. $\left.\begin{array}{l}\text{If } (a^2-b^2)(3x+5y)=(4a-b)2ab\ldots\ldots(1),\\ \text{and } a^2x-\dfrac{ab^2c}{a+b}+(a+b+c)by=b^2x+(a+2b)ab\ldots\ldots.(2),\end{array}\right\}$ find x and y.

From (1) $\quad (a^2-b^2)3x+(a^2-b^2)5y=8a^2b-2ab^2$,

...... (2) $\quad (a^2-b^2)3x+3(a+b+c)by=3a^2b+6ab^2+\dfrac{3ab^2c}{a+b}$,

$$\begin{aligned}\therefore\ (5a^2-8b^2-3ab-3bc)y &= 5a^2b-8ab^2-\frac{3ab^2c}{a+b},\\ &= ab\left\{5a-8b-\frac{3bc}{a+b}\right\},\\ &= \frac{ab}{a+b}\{5a^2-8b^2-3ab-3bc\};\end{aligned}$$

$$\therefore\ y=\frac{ab}{a+b}.$$

And $\therefore$ substituting in (1), $(a^2-b^2)3x+(a-b)5ab=8a^2b-2ab^2$,

$$\begin{aligned}(a^2-b^2)3x &= 8a^2b-2ab^2-5a^2b+5ab^2,\\ &= 3a^2b+3ab^2=3ab(a+b),\end{aligned}$$

$$\therefore\ x=\frac{ab}{a-b}.$$

Ex. 30. $\left.\begin{array}{l}\text{If } \sqrt{x+y}+\sqrt{x-y}=\dfrac{1}{a}(xy-y\sqrt{x^2-y^2}),\\ \sqrt[4]{x+y}+\sqrt[4]{x-y}=b,\end{array}\right\}$ find x and y.

From 1st. $\sqrt{x+y}+\sqrt{x-y}=\dfrac{y}{2a}(2x-2\sqrt{x^2-y^2})$,

$$=\frac{y}{2a}(\sqrt{x+y}-\sqrt{x-y})^2,$$

$$\frac{y}{2a}(\sqrt{x+y}-\sqrt{x-y})^3=(\sqrt{x+y}+\sqrt{x-y})(\sqrt{x+y}-\sqrt{x-y}),$$

$$=2y,$$

$$\sqrt{x+y}-\sqrt{x-y}=\sqrt[3]{4a},$$

or $(\sqrt[4]{x+y}+\sqrt[4]{x-y})(\sqrt[4]{x+y}-\sqrt[4]{x-y})=\sqrt[3]{4a}$,

$$\left.\begin{aligned}\therefore\ \sqrt[4]{x+y}-\sqrt[4]{x-y}&=\frac{\sqrt[3]{4a}}{b},\\ \text{and}\ \sqrt[4]{x+y}+\sqrt[4]{x-y}&=b,\end{aligned}\right\}$$

$$\therefore\ \sqrt[4]{x+y}=\frac{1}{2}\left(b+\frac{\sqrt[3]{4a}}{b}\right),\ \text{and}\ \sqrt[4]{x-y}=\frac{1}{2}\left(b-\frac{\sqrt[3]{4a}}{b}\right).$$

$$x+y=\frac{1}{16}\left\{b^4+4b^2\sqrt[3]{4a}+6\sqrt[3]{16a^2}+\frac{16a}{b^2}+\frac{\sqrt[3]{(4a)^4}}{b^4}\right\},$$

$$x-y=\frac{1}{16}\left\{b^4-4b^2\sqrt[3]{4a}+6\sqrt[3]{16a^2}-\frac{16a}{b^2}+\frac{\sqrt[3]{(4a)^4}}{b^4}\right\},$$

$$\therefore\ x=\frac{1}{16}\left\{b^4+12\sqrt[3]{2a^2}+\frac{\sqrt[3]{(4a)^4}}{b^4}\right\},$$

$$=\frac{1}{4}\left\{\frac{b^2}{2}+\frac{\sqrt[3]{2a^2}}{b^2}\right\}^2+\frac{1}{2}\sqrt[3]{2a^2},$$

$$\text{and}\ y=\frac{1}{16}\left\{4b^2\sqrt[3]{4a}+16\frac{a}{b^2}\right\}=\frac{b^2}{2}\cdot\sqrt[3]{\frac{a}{2}}+\frac{a}{b^2}.$$

SIMPLE EQUATIONS OF THREE UNKNOWN QUANTITIES.

Ex. 6. If $4x-5y+mz=7x-11y+nz=x+y+pz=3$, find x, y, z.

$x+y+pz=3$; $\quad\therefore\ 4x+4y+4pz=12$,

and $4x-5y+mz=3$,

$\therefore\ 9y+(4p-m)z=9$.........(1).

$$\left.\begin{aligned}\text{Again, }\ 7x+7y+7pz&=21,\\ \text{and }\ 7x-11y+nz&=3,\end{aligned}\right\}$$

$$\begin{aligned}&\therefore\ 18y+(7p-n)z=18.\\ \text{But from (1)}\ \ &18y+(8p-2m)z=18,\end{aligned}\Bigg\}\cdots\cdots\cdots(2).$$

$$\therefore\ (p-2m+n)z=0,$$

$$\left.\begin{aligned}&\therefore\ z=0.\\ \text{Hence } 18y=18;\ \ &\therefore\ y=1,\\ \text{and } x+1=3;\ \ &\therefore\ x=2.\end{aligned}\right\}$$

Ex. 14. If $\left.\begin{aligned}&x+y+z=0\cdots\cdots\cdots\cdots\cdots\cdots(1),\\ &(a+b)x+(a+c)y+(b+c)z=0\ldots(2),\\ &abx+acy+bcz=1\cdots\cdots\cdots\cdots(3).\end{aligned}\right\}$ find x, y, and z.

$$\left.\begin{aligned}\text{From (1)}\quad &(b+c)x+(b+c)y+(b+c)z=0,\\ \ldots\ \ (2)\quad &(a+b)x+(a+c)y+(b+c)z=0,\end{aligned}\right\}$$

$$\therefore\ (a-c)x+(a-b)y=0,\ldots\ldots(4)$$

$$\left.\begin{aligned}\text{Again, from (1)}\quad &bcx+bcy+bcz=0,\\ \ldots\ \ (3)\quad &abx+acy+bcz=1,\end{aligned}\right\}$$

$$\therefore\ (ab-bc)x+(ac-bc)y=1,$$

$$\left.\begin{aligned}\text{or }\ &b(a-c)x+c(a-b)y=1,\\ \text{from (4)}\ \ &c(a-c)x+c(a-b)y=0,\end{aligned}\right\}$$

$$\therefore\ (a-c)(b-c)x=1,\quad \therefore\ x=\frac{1}{(a-c)(b-c)}.$$

Also $y=-\dfrac{a-c}{a-b}.x=\dfrac{-1}{(a-b)(b-c)}$.

And $z=-x-y=\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(a-c)(b-c)}=\dfrac{1}{(a-b)(a-c)}$.

Ex. 15. If $\left.\begin{aligned}&x-ay+a^2z-a^3=0\ldots(1),\\ &x-by+b^2z-b^3=0\ldots(2),\\ &x-cy+c^2z-c^3=0\ldots(3).\end{aligned}\right\}$ find x, y, and z.

From (1) and (2) $(a-b)y-(a^2-b^2)z+a^3-b^3=0$,

$$\left.\begin{aligned}\text{or }\ &y-(a+b)z+a^2+ab+b^2=0,\\ \text{Similarly, from (2) and (3)}\ \ &y-(b+c)z+b^2+bc+c^2=0,\end{aligned}\right\}$$

$$\therefore\ (a-c)z-(a^2+ab-bc-c^2)=0,$$

$$(a-c)z=a^2-c^2+(a-c)b,$$

$$\therefore\ z=a+c+b,\ \text{ or }\ a+b+c.$$

Ex. 82. If $\frac{ax-b^2}{\sqrt{ax}+b}=\frac{\sqrt{ax}-b}{n}-c$, find x.

$$\sqrt{ax}-b=\frac{\sqrt{ax}-b}{n}-c,$$

$$(\sqrt{ax}-b)\left(1-\frac{1}{n}\right)=-c,$$

$$\sqrt{ax}-b=-\frac{nc}{n-1},$$

$$ax=\left(b-\frac{nc}{n-1}\right)^2,$$

$$\therefore\ x=\frac{1}{a}\left(b-\frac{nc}{n-1}\right)^2.$$

Ex. 85. If $\sqrt[m]{a+x}=\sqrt[2m]{x^2+8ax+b^2}$, find x.

Raising both sides to the $2m^{\text{th}}$ power,

$$(a+x)^2=x^2+8ax+b^2,$$

$$a^2+2ax=8ax+b^2,$$

$$6ax=a^2-b^2,$$

$$\therefore\ x=\frac{a^2-b^2}{6a}.$$

Ex. 87. This is similar to Ex. 82.

Ex. 90. If $\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}=1$, find x.

$$\sqrt{\frac{1+bx}{1-bx}}=\frac{1+ax}{1-ax},$$

$$\frac{1+bx}{1-bx}=\frac{1+2ax+a^2x^2}{1-2ax+a^2x^2},$$

$$\frac{bx}{1}=\frac{2ax}{1+a^2x^2},\ \text{(Art. 195)}.$$

$$1+a^2x^2=\frac{2a}{b},$$

$$a^2x^2=\frac{2a}{b}-1,$$

$$\therefore\ x=\frac{1}{a}\sqrt{\frac{2a}{b}-1}.$$

Ex. 91. If $\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{62}{63}\cdot\dfrac{1+x}{1-x}$, find x.

$$\frac{62}{63}=\frac{(1-x)(1+x+x^2)}{(1+x)(1-x+x^2)}=\frac{1-x^3}{1+x^3},$$

$$\frac{2x^3}{2}=\frac{63-62}{63+62},$$

$$\text{or } x^3=\frac{1}{125}; \quad \therefore x=\sqrt[3]{\frac{1}{125}}=\frac{1}{5}.$$

Ex. 92. If $\left(\dfrac{a+x}{a-x}\right)^2=1+\dfrac{cx}{ab}$, find x.

$$1+\frac{cx}{ab}=\left(1+\frac{2x}{a-x}\right)^2,$$

$$=1+\frac{4x}{a-x}+\frac{4x^2}{(a-x)^2};$$

$$\therefore \frac{c}{ab}=\frac{4}{a-x}+\frac{4x}{(a-x)^2},$$

$$=\frac{4a}{(a-x)^2},$$

$$\frac{c}{b}=\frac{4a^2}{(a-x)^2},$$

$$\therefore a-x=\pm 2a\sqrt{\frac{b}{c}},$$

$$\therefore x=a\left\{1\mp 2\sqrt{\frac{b}{c}}\right\}.$$

Ex. 96. If $a+b=\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, find x.

$$(a+b)x+(a+b)\sqrt{1+x^2}=2a\sqrt{1+x^2},$$

$$(a+b)x=(a-b)\sqrt{1+x^2},$$

$$\frac{\sqrt{1+x^2}}{x}=\frac{a+b}{a-b},$$

$$\frac{1}{x^2}+1=\left(\frac{a+b}{a-b}\right)^2,$$

$$\frac{1}{x^2}=\left(\frac{a+b}{a-b}\right)^2-1=\frac{4ab}{(a-b)^2},$$

$$\therefore x=\frac{a-b}{2\sqrt{ab}}=\frac{1}{2}\left\{\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right\}.$$

$$x^2-\frac{16a}{3}\cdot x+\left(\frac{8a}{3}\right)^2=4a^2+\frac{64a^2}{9}=\frac{100a^2}{9};$$

$$\therefore\ x=\frac{8a\pm10a}{3}=6a,\ \text{ or }\ -\frac{2a}{3}.$$

Ex. 42. If $\left(x-\frac{ab}{x}\right)^2=\frac{a}{2}(a+b)\left(1+\frac{a^2}{x^2}\right)$, find x.

$$(x^2-ab)^2=\frac{a}{2}(a+b)(x^2+a^2),$$

$$(x^2-ab)^2-\frac{a}{2}(a+b)(x^2-ab)=\frac{a}{2}(a+b)(a^2+ab),$$

$$=\frac{a^2}{2}(a+b)^2,$$

$$(x^2-ab)^2-\frac{a}{2}(a+b)(x^2-ab)+\frac{a^2}{16}(a+b)^2=\frac{a^2}{16}(a+b)^2+\frac{a^2}{2}(a+b)^2,$$

$$=\frac{a^2}{16}\cdot9(a+b)^2;$$

$$\therefore\ x^2-ab-\frac{a}{4}(a+b)=\pm\frac{a}{4}\cdot3(a+b),$$

$$x^2=ab+\frac{a}{4}\{a+b\pm3(a+b)\},$$

$$=2ab+a^2,\ \text{ or }\ \frac{a}{2}(b-a);$$

$$\therefore\ x=\sqrt{2ab+a^2},\ \text{ or }\ \sqrt{\frac{a}{2}(b-a)}.$$

Ex. 59. If $\sqrt{x^2+1}-\sqrt{x^2-1}=\frac{\sqrt{2x}}{\sqrt[8]{x^4+1}}$, find x.

$$2=\frac{\sqrt{2x}}{\sqrt[8]{x^4+1}}\cdot\{\sqrt{x^2+1}+\sqrt{x^2-1}\},$$

$$4=\frac{2x}{\sqrt[4]{x^4+1}}\cdot\{2x^2+2\sqrt{x^4-1}\},$$

$$\sqrt[4]{1+\frac{1}{x^4}}=x^2\cdot\left\{1+\sqrt{1-\frac{1}{x^4}}\right\},$$

$$\sqrt{1+\frac{1}{x^4}}=x^4\cdot\left\{2-\frac{1}{x^4}+2\sqrt{1-\frac{1}{x^4}}\right\},$$

$$=2x^4-1+2x^4\sqrt{1-\frac{1}{x^4}},$$

$$1+\sqrt{1+\frac{1}{x^4}}=2x^4\left\{1+\sqrt{1-\frac{1}{x^4}}\right\},$$

$$\left(1+\sqrt{1+\frac{1}{x^4}}\right)\left(1-\sqrt{1-\frac{1}{x^4}}\right)=2,$$

$$1+\sqrt{1+\frac{1}{x^4}}-\sqrt{1-\frac{1}{x^4}}-\sqrt{1-\frac{1}{x^8}}=2,$$

$$\sqrt{1+\frac{1}{x^4}}-\sqrt{1-\frac{1}{x^4}}=1+\sqrt{1-\frac{1}{x^8}},$$

$$2-2\sqrt{1-\frac{1}{x^8}}=2-\frac{1}{x^8}+2\sqrt{1-\frac{1}{x^8}},$$

$$\left(1-\frac{1}{x^8}\right)+4\sqrt{1-\frac{1}{x^8}}+4=5,$$

$$\sqrt{1-\frac{1}{x^8}}=\pm\sqrt{5}-2,$$

$$1-\frac{1}{x^8}=9\mp4\sqrt{5},$$

$$\frac{1}{x^8}=\pm4\sqrt{5}-8,$$

$$\therefore\ x=\sqrt{\frac{1}{\pm4\sqrt{5}-8}}=\sqrt[8]{\frac{2\pm\sqrt{5}}{4}}.$$

Ex. 65. If $\dfrac{2x+\sqrt{2(1+x)}}{1-x}=a+\dfrac{1}{a}$, find x.

For $a+\dfrac{1}{a}$ write p, then

$$2x+\sqrt{2(1+x)}=p-px,$$

$$\sqrt{2(1+x)}=p-(p+2)x,$$

$$2+2x=p^2-2p(p+2)x+(p+2)^2x^2.$$

$$(p+2)^2x^2-2(p+1)^2x=2-p^2,$$

$$x^2-2\left(\frac{p+1}{p+2}\right)^2x+\left(\frac{p+1}{p+2}\right)^4=\left(\frac{p+1}{p+2}\right)^4+\frac{2-p^2}{(p+2)^2},$$

$$=\frac{4p^2+12p+9}{(p+2)^4},$$

$$\therefore\ x=\frac{(p+1)^2\pm(2p+3)}{(p+2)^2},$$

$$x=1, \text{ or } \frac{p^2-2}{(p+2)^2},$$

$$=1, \text{ or } \frac{a^2+\frac{1}{a^2}}{\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)^4} \because p+2=\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)^2.$$

Ex. 66. If $(a+b)\sqrt{a^2+b^2+x^2}-(a-b)\sqrt{a^2+b^2-x^2}=a^2+b^2$, find x.

$$\left.\begin{array}{l}(a^2+b^2+2ab)(a^2+b^2+x^2)\\+(a^2+b^2-2ab)(a^2+b^2-x^2)\end{array}\right\}-2(a^2-b^2)\sqrt{(a^2+b^2)^2-x^4}=(a^2+b^2)^2,$$

$$2(a^2+b^2)^2+4abx^2-2(a^2-b^2)\sqrt{(a^2+b^2)^2-x^4}=(a^2+b^2)^2,$$

$$(a^2+b^2)^2+4abx^2=2(a^2-b^2)\sqrt{(a^2+b^2)^2-x^4},$$

$$(a^2+b^2)^4+8ab(a^2+b^2)^2x^2+16a^2b^2x^4=4(a^2-b^2)^2(a^2+b^2)^2-4(a^2-b^2)^2x^4,$$

$$4(a^2+b^2)^2x^4+8ab(a^2+b^2)^2x^2=(a^2+b^2)^2\{4(a^2-b^2)^2-(a^2+b^2)^2\},$$

$$4x^4+8abx^2=3a^4-10a^2b^2+3b^4,$$

$$4x^4+8abx^2+4a^2b^2=3a^4-6a^2b^2+3b^4,$$

$$2x^2+2ab=\pm\sqrt{3}(a^2-b^2),$$

$$x^2=\pm\sqrt{\frac{3}{2}}(a^2-b^2)-ab,$$

$$\therefore\ x=\sqrt{\pm\sqrt{\frac{3}{2}}(a^2-b^2)-ab}.$$

Ex. 67. If $\frac{1}{ax}\left(\frac{a^2+x^2}{5}\right)^2+12\frac{2}{3}\left(\frac{ax}{5}\right)^2=\frac{(a^2+x^2)^2}{6}$, find x.

$$6\times12\tfrac{2}{3}=25\left(\frac{a^2+x^2}{ax}\right)^2-6\frac{(a^3+x^3)^2}{a^3x^3},$$

$$76=25\left(\frac{a}{x}+\frac{x}{a}\right)^2-6\left(\frac{a^3}{x^3}+\frac{x^3}{a^3}+2\right).$$

For $\frac{a}{x}+\frac{x}{a}$ write y, then $\frac{a^3}{x^3}+\frac{x^3}{a^3}=y^3-3y$,

$$\therefore\ 6y^3-18y-25y^2+88=0,$$

$$6y^2(y-4)-y(y-4)-22(y-4)=0,$$

$$\therefore\ y-4=0, \text{ or } 6y^2-y-22=0,$$

$$y=4, \text{ or } y^2-\frac{1}{6}y+\left(\frac{1}{12}\right)^2=\frac{1}{144}+\frac{22}{6}=\frac{529}{(12)^2},$$

$$y=\frac{1+23}{12}=2.$$

$$\therefore \frac{a}{x}+\frac{x}{a}=4, \qquad \text{or } \frac{a}{x}+\frac{x}{a}=2,$$

$$x^2-4ax+4a^2=3a^2, \qquad x^2-2ax+a^2=0,$$

$$\therefore x=(2\pm\sqrt{3}).a. \qquad \therefore x=a.$$

Ex. 68. If $\sqrt{(b^2+c^2)(a^2+x^2)}=n(ac+bx)$, find x.

$$(b^2+c^2)\left(1+\frac{x^2}{a^2}\right)=n^2\left(c+b\frac{x}{a}\right)^2,$$

$$=n^2\left(c^2+2bc.\frac{x}{a}+b^2.\frac{x^2}{a^2}\right),$$

$$\{c^2-(n^2-1)b^2\}\frac{x^2}{a^2}-2n^2bc.\frac{x}{a}=-\{b^2-(n^2-1)c^2\},$$

$$\frac{x^2}{a^2}-\frac{2n^2bc}{c^2-(n^2-1)b^2}.\frac{x}{a}+\left\{\frac{n^2bc}{c^2-(n^2-1)b^2}\right\}^2=\frac{n^4b^2c^2-\{b^2-(n^2-1)c^2\}\{c^2-(n^2-1)b^2\}}{\{c^2-(n^2-1)b^2\}^2},$$

$$=\frac{(n^2-1)(b^4+2b^2c^2+c^4)}{\{c^2-(n^2-1)b^2\}^2},$$

$$\frac{x}{a}=\frac{n^2bc\pm\sqrt{n^2-1}.(b^2+c^2)}{c^2-(n^2-1)b^2},$$

$$=\frac{(b\mp\sqrt{n^2-1}.c)(c\mp\sqrt{n^2-1}.b)}{(c\pm\sqrt{n^2-1}.b)(c\mp\sqrt{n^2-1}.b)}=\frac{b\mp\sqrt{n^2-1}.c}{c\pm\sqrt{n^2-1}.b},$$

$$\therefore x=\frac{b\mp\sqrt{n^2-1}.c}{c\pm\sqrt{n^2-1}.b}.a.$$

Ex. 70. If $\sqrt[5]{(a+x)^2}+\sqrt[5]{(a-x)^2}=3\sqrt[5]{a^2-x^2}$, find x.

Divide by $\sqrt[5]{(a+x)^2}$, then $1+\left(\frac{a-x}{a+x}\right)^{\frac{2}{5}}=3.\left(\frac{a-x}{a+x}\right)^{\frac{1}{5}}$;

for $\left(\frac{a-x}{a+x}\right)^{\frac{1}{5}}$ write y, then $y^2-3y=-1$,

$$y^2-3y+\left(\frac{3}{2}\right)^2=\frac{9}{4}-1=\frac{5}{4},$$

$$\therefore y=\frac{3\pm\sqrt{5}}{2},$$

$$\text{or } \frac{a-x}{a+x}=\frac{(3\pm\sqrt{5})^5}{2^5}, \therefore$$

$$\therefore \frac{x}{a}=\frac{(3\pm\sqrt{5})^5-2^5}{(3\pm\sqrt{5})^5+2^5},$$

$$\therefore x=\frac{(3\pm\sqrt{5})^5-2^5}{(3\pm\sqrt{5})^5+2^5}.a=\pm\frac{11}{25}\sqrt{5}.a.$$

Ex. 71. If $\dfrac{(27a+8x)^{\frac{2}{3}}}{15x^{\frac{13}{15}}}+\dfrac{8x^{\frac{2}{15}}}{3\sqrt[3]{27a+8x}}=\dfrac{8}{5\sqrt[5]{x}}$, find x.

$$27a+8x+40x=24x^{\frac{2}{3}}(27a+8x)^{\frac{1}{3}},$$

$$27a+48x=24x^{\frac{2}{3}}(27a+8x)^{\frac{1}{3}},$$

$$9a+16x=8x^{\frac{2}{3}}(27a+8x)^{\frac{1}{3}},$$

for $9a$ write b, and for $8x$ write y, then

$$b+2y=2y^{\frac{2}{3}}(3b+y)^{\frac{1}{3}},$$

$$\therefore b^3+6b^2y+12by^2+8y^3=24by^2+8y^3,$$

$$12by^2-6b^2y=b^3,$$

$$y^2-\frac{b}{2}y=\frac{b^2}{12},$$

$$y^2-\frac{b}{2}y+\left(\frac{b}{4}\right)^2=\frac{b^2}{12}+\frac{b^2}{16}=\frac{7b^2}{48},$$

$$y=\frac{b}{4}\left\{1\pm\sqrt{\frac{7}{3}}\right\}=\frac{b}{12}(3\pm\sqrt{21})=\frac{3a}{4}(3\pm\sqrt{21});$$

$$\therefore x=\frac{y}{8}=\frac{3}{32}(3\pm\sqrt{21}).a.$$

Ex. 72. If $a^2b^2x^{\frac{1}{q}}-4(ab)^{\frac{3}{2}}.x^{\frac{p+q}{2pq}}=(a-b)^2.x^{\frac{1}{p}}$, find x.

$$a^2b^2.x^{\frac{1}{q}-\frac{1}{p}}-4(ab)^{\frac{3}{2}}.x^{\frac{p+q}{2pq}-\frac{1}{p}}=(a-b)^2,$$

$$a^2b^2.x^{\frac{p-q}{pq}}-4(ab)^{\frac{1}{2}}.ab.x^{\frac{p-q}{2pq}}=(a-b)^2,$$

$$(ab)^2.x^{\frac{p-q}{pq}}-4(ab)^{\frac{1}{2}}.ab.x^{\frac{p-q}{2pq}}+4ab=(a-b)^2+4ab=(a+b)^2;$$

$$\therefore ab.x^{\frac{p-q}{2pq}}=2\sqrt{ab}\pm(a+b),$$

$$=(\sqrt{a}+\sqrt{b})^2,\quad \text{or } -(\sqrt{a}-\sqrt{b})^2;$$

$$x^{\frac{p-q}{2pq}}=\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)^2,\quad \text{or } -\left(\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}\right)^2;$$

$$\therefore x=\left(\frac{1}{\sqrt{b}}\pm\frac{1}{\sqrt{a}}\right)^{\frac{4pq}{p-q}}.$$

Ex. 73. If $\sqrt[2pq]{x^{p+q}}-\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot(\sqrt[p]{x}+\sqrt[q]{x})=0$, find x.

$$x^{\frac{p+q}{2pq}-\frac{1}{q}}-\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot x^{\frac{1}{p}-\frac{1}{q}}-\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}=0,$$

$$x^{\frac{q-p}{2pq}}-\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot x^{\frac{q-p}{pq}}=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2};$$

$$\therefore\ x^{\frac{q-p}{pq}}-2\cdot\frac{a^2+b^2}{a^2-b^2}\cdot x^{\frac{q-p}{2pq}}+\left(\frac{a^2+b^2}{a^2-b^2}\right)^2=\left(\frac{a^2+b^2}{a^2-b^2}\right)^2-1,$$

$$=\frac{4a^2b^2}{(a^2-b^2)^2};$$

$$\therefore\ x^{\frac{q-p}{2pq}}=\frac{a^2+b^2}{a^2-b^2}\pm\frac{2ab}{a^2-b^2}=\frac{(a\pm b)^2}{a^2-b^2}=\frac{a\pm b}{a\mp b};$$

$$\therefore\ x=\left(\frac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}.$$

QUADRATIC EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

Ex. 13. If $x+y=5$, and $(x^2+y^2)(x^3+y^3)=455$, find x and y.

$\div$ 2nd by 1st $\quad (x^2+y^2)(x^2-xy+y^2)=91,$

$$(x^2+y^2)^2-(x^2+y^2)xy=91.$$

But from 1st $\quad x^2+y^2=25-2xy,$

$$\therefore\ (25-2xy)^2-(25-2xy)xy=91,$$

$$625-100xy+4(xy)^2-25xy+2(xy)^2=91,$$

$$6(xy)^2-125xy=-534,$$

$$(xy)^2-\frac{125}{6}xy+\left(\frac{125}{12}\right)^2=\frac{15625}{(12)^2}-\frac{534}{6}=\frac{2809}{(12)^2},$$

$$\therefore\ xy=\frac{125\pm53}{12}=14\tfrac{5}{6},\text{ or }6.$$

Taking the latter value for xy, we have

$$\left.\begin{aligned}x^2+2xy+y^2&=25,\\4xy&=24,\end{aligned}\right\}$$

$$\therefore\ x^2-2xy+y^2=1,$$

$$\left.\begin{aligned} x-y&=\pm 1,\\ \text{and } x+y&=\ \ 5,\end{aligned}\right\}$$

$$\therefore\ 2x=6, \text{ or } 4, \quad \therefore\ x=3, \text{ or } 2.$$

$$2y=4, \text{ or } 6, \quad \therefore\ y=2, \text{ or } 3.$$

Ex. 18. If $\sqrt{ax}+\sqrt{by}=\frac{1}{2}(x+y)=a+b$, find x and y.

$$\left.\begin{aligned} ax+by+2\sqrt{abxy}&=(a+b)^2,\\ \text{also } bx+by\qquad\qquad &=2ab+2b^2,\end{aligned}\right\}$$

$$\therefore\ (a-b)x+2\sqrt{abxy}=a^2-b^2,$$

$$\therefore\ 4abxy=\{a^2-b^2-(a-b)x\}^2.$$

$$\text{But } y=2(a+b)-x,$$

$$\therefore\ 4abx\{2(a+b)-x\}=\{a^2-b^2-(a-b)x\}^2,$$

$$=(a-b)^2.(a+b-x)^2,$$

$$4ab\{2(a+b)x-x^2\}=(a-b)^2\{\overline{a+b}|^2-2(a+b)x+x^2\},$$

$$\{2(a+b)x-x^2\}.\{4ab+(a-b)^2\}=(a^2-b^2)^2,$$

$$x^2-2(a+b)x=-\frac{(a^2-b^2)^2}{(a+b)^2}=-(a-b)^2,$$

$$x^2-2(a+b)x+(a+b)^2=(a+b)^2-(a-b)^2=4ab,$$

$$x=a+b\pm 2\sqrt{ab}=(\sqrt{a}\pm\sqrt{b})^2.$$

$$\text{And } y=2(a+b)-x=a+b\mp 2\sqrt{ab}=(\sqrt{a}\mp\sqrt{b})^2.$$

Ex. 19. If $xz=y^2$, $(x+y)(z-x-y)=3$, and $(x+y+z)\times(z-x-y)=7$, find x, y, z.

Divide 3rd equation by 2nd,

$$\frac{x+y+z}{x+y}=\frac{7}{3},$$

$$1+\frac{z}{x+y}=\frac{7}{3},$$

$$\frac{z}{x+y}=\frac{7}{3}-1=\frac{4}{3},$$

$$\therefore\ x+y=\frac{3}{4}z.$$

Substituting this value in 2nd equation,

$$\frac{3}{4}z\left(z-\frac{3}{4}z\right)=3,$$

$$\frac{3z^2}{16}=3; \quad \therefore\ z=4.$$

$$\text{But } y^2=xz=4x,$$

$$\therefore\ y=2\sqrt{x},$$

$$\text{and } x+y=\frac{3}{4}z=3,$$

$$\therefore\ x+2\sqrt{x}=3,$$

$$x+2\sqrt{x}+1=4,$$

$$\sqrt{x}=\pm 2-1=1, \text{ or } -3,$$

$$\therefore\ x=1, \text{ or } 9.$$

$$\text{and } y=2\sqrt{x}=2, \text{ or } -6.$$

Ex. 40. If $\left.\begin{array}{l}(\sqrt{x})^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{y})^{\frac{8}{3}},\ldots\ldots(1)\\ \text{and } (\sqrt{y})^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{x})^{\frac{2}{3}},\ldots\ldots(2)\end{array}\right\}$ find x and y.

To save writing, put x for $\sqrt{x}$, and y for $\sqrt{y}$, then

$$x^{\sqrt{x}+\sqrt{y}}=y^{\frac{8}{3}}, \text{ and } y^{\sqrt{x}+\sqrt{y}}=x^{\frac{2}{3}},$$

$$\therefore\ y=x^{\frac{2}{3(\sqrt{x}+\sqrt{y})}}, \text{ and } x^{\sqrt{x}+\sqrt{y}}=y^{\frac{8}{3}}=x^{\frac{16}{9(\sqrt{x}+\sqrt{y})}},$$

$$\therefore\ \sqrt{x}+\sqrt{y}=\frac{16}{9(\sqrt{x}+\sqrt{y})},$$

$$\therefore\ (\sqrt{x}+\sqrt{y})^2=\frac{16}{9},$$

$$\sqrt{x}+\sqrt{y}=\frac{4}{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence, $x^{\frac{4}{3}}=y^{\frac{8}{3}}$, and $\therefore\ x=y^2$, and $\sqrt{x}=y$,

$$\therefore \text{ from (3) } y+\sqrt{y}=\frac{4}{3},$$

$$y+\sqrt{y}+\frac{1}{4}=\frac{4}{3}+\frac{1}{4}=\frac{19}{12}=\frac{57}{36},$$

$$\therefore\ \sqrt{y}=-\frac{1}{2}\pm\frac{1}{6}\sqrt{57},$$

$$\therefore\ y=\left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^2,$$

$$\text{and}\ \ x=y^2=\left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^4.$$

But x and y were written for $\sqrt{x}$ and $\sqrt{y}$, therefore the values required are

$$x=\left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^8,$$

$$\text{and}\ y=\left\{-\frac{1}{2}\pm\frac{1}{6}\sqrt{57}\right\}^4.$$

Ex. 43. If $\left.\begin{array}{l}\sqrt{x^2+\sqrt[3]{x^4y^2}}+\sqrt{y^2+\sqrt[3]{y^4x^2}}=a\ldots\ldots(1)\\ \text{and}\ \ x+y+3\sqrt[3]{bxy}=b\ldots\ldots(2)\end{array}\right\}$ find x and y.

From (1) $\sqrt{x^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})}+\sqrt{y^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})}=a,$

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{1}{2}}=a,$$

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=a,$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) since $(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3=x+y+3\sqrt[3]{xy}.\{x^{\frac{1}{3}}+y^{\frac{1}{3}}\},$

$$\therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=b^{\frac{1}{3}},$$

$$\text{and}\ \ x^{\frac{2}{3}}+y^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}=b^{\frac{2}{3}},$$

$$\therefore\ 2x^{\frac{1}{3}}y^{\frac{1}{3}}=b^{\frac{2}{3}}-a^{\frac{2}{3}},$$

$$\therefore\ x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=2a^{\frac{2}{3}}-b^{\frac{2}{3}},$$

$$x^{\frac{1}{3}}-y^{\frac{1}{3}}=\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}},$$

$$2x^{\frac{1}{3}}=b^{\frac{1}{3}}+\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}},$$

$$\therefore\ x=\frac{1}{8}\left\{b^{\frac{1}{3}}+\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\right\}^3,$$

$$\text{and}\ y=\frac{1}{8}\left\{b^{\frac{1}{3}}-\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\right\}^3.$$

Ex. 44. If $\left.\begin{array}{l}(x^2+y^2)\dfrac{y}{x}=8\frac{2}{3}, \\ (x^2-y^2)\dfrac{x}{y}=7\frac{1}{2},\end{array}\right\}$ find x and y.

$$\frac{x^2+y^2}{x^2-y^2}\cdot\frac{y^2}{x^2}=\frac{8\frac{2}{3}}{7\frac{1}{2}}=\frac{52}{45},$$

$$\frac{\dfrac{x^2}{y^2}+1}{\dfrac{x^2}{y^2}-1}=\frac{52}{45}\cdot\frac{x^2}{y^2},$$

$$\frac{45}{52}\cdot\frac{x^2}{y^2}+\frac{45}{52}=\frac{x^4}{y^4}-\frac{x^2}{y^2},$$

$$\frac{x^4}{y^4}-\frac{97}{52}\cdot\frac{x^2}{y^2}+\left(\frac{97}{104}\right)^2=\frac{18769}{(104)^2},$$

$$\therefore \frac{x^2}{y^2}=\frac{97\pm137}{104}=\frac{234}{104}=\frac{9}{4},$$

$$\therefore \frac{x}{y}=\frac{3}{2}, \quad \text{and } y=\frac{2x}{3}.$$

But $x^4-y^4=8\frac{2}{3}\times7\frac{1}{2}=65$, multiplying 1st equation by 2nd,

$$\therefore x^4-\left(\frac{2x}{3}\right)^4=65,$$

$$65x^4=81\times65,$$

$$\therefore x^4=81, \text{ and } x=\pm3. \quad \text{Also } y=\frac{2x}{3}=\pm2.$$

Ex. 45. If $\left.\begin{array}{l}(x^2+y^2)(x+y)\;=15xy, \\ (x^4+y^4)(x^2+y^2)=85x^2y^2,\end{array}\right\}$ find x and y.

Squaring 1st, $(x^4+y^4)(x+y)^2+2x^2y^2(x+y)^2=225x^2y^2$,

$$(x^4+y^4)(x^2+y^2)+2xy(x^4+y^4)+2x^2y^2(x+y)^2=225x^2y^2,$$

$\therefore$ from 2nd, $85x^2y^2+2xy(x^4+y^4)+2x^2y^2(x+y)^2=225x^2y^2$,

$$2(x^4+y^4)+2xy(x+y)^2=140xy,$$

$$x^4+y^4+xy(x+y)^2=70xy,$$

$$(x+y)^2+\frac{x^4+y^4}{xy}=70\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now dividing the 2nd equation by the 1st, $\dfrac{x^4+y^4}{x+y}=\dfrac{85}{15}xy$,

$$\therefore \frac{x^4+y^4}{xy}=\frac{85}{15}(x+y),$$

and substituting in (1), $(x+y)^2+\frac{85}{15}(x+y)=70$,

$$(x+y)^2+\frac{85}{15}(x+y)+\left(\frac{85}{30}\right)^2=\frac{7225}{(30)^2}+70=\frac{70225}{(30)^2},$$

$$\therefore x+y=\frac{\pm 265-85}{30}=6, \text{ or } -11\tfrac{2}{3}\ldots\ldots(2).$$

$\therefore$ From 1st, $x^2+y^2=\frac{15xy}{6}$, taking 1st value of $x+y$,

$$x^2-\frac{15y}{6}x+\left(\frac{15y}{12}\right)^2=\frac{225y^2}{144}-y^2=\frac{81y^2}{144},$$

$$\therefore x=\frac{15\pm 9}{12}.y=2y, \text{ or } \frac{1}{2}y,$$

and $x+y=6$, from (2)

$2x+y=6$,

$3y=6$,

$\left.\begin{array}{r} y=2, \\ \text{and } \therefore x=4. \end{array}\right\}$

or $x+y=6$,

$\frac{1}{2}y+y=6$,

$3y=12$,

$\left.\begin{array}{r} y=4, \\ \text{and } \therefore x=2. \end{array}\right\}$

Ex. 48. If $\left.\begin{array}{l} \sqrt[3]{x+y}+\sqrt[3]{x-y}=\sqrt[3]{a}, \\ \sqrt[3]{x^2+y^2}+\sqrt[3]{x^2-y^2}=\sqrt[3]{a^2}, \end{array}\right\}$ find x and y.

From 1st, cubing, $2x+3(\sqrt[3]{x+y}+\sqrt[3]{x-y})\sqrt[3]{x^2-y^2}=a$,

$$2x+3\sqrt[3]{a}.\sqrt[3]{x^2-y^2}=a,$$

$$\therefore \sqrt[3]{x^2-y^2}=\frac{a-2x}{3\sqrt[3]{a}}\ldots(1).$$

From 2nd, similarly, $\sqrt[3]{x^2+y^2}.\sqrt[3]{x^2-y^2}=\frac{a^2-2x^2}{3\sqrt[3]{a^2}}$,

$$\therefore \text{by (1)}, \sqrt[3]{x^2+y^2}.\frac{a-2x}{3\sqrt[3]{a}}=\frac{a^2-2x^2}{3\sqrt[3]{a^2}},$$

$$\therefore \sqrt[3]{x^2+y^2}=\frac{a^2-2x^2}{(a-2x)\sqrt[3]{a}},\ldots\ldots\ldots(2),$$

$\therefore$ from 2nd, by (1) and (2), $\dfrac{a^2-2x^2}{(a-2x)\sqrt[3]{a}}+\dfrac{a-2x}{3\sqrt[3]{a}}=\sqrt[3]{a^2}$,

$$3(a^2-2x^2)+(a-2x)^2=3a(a-2x),$$

$$3a^2-6x^2+a^2-4ax+4x^2=3a^2-6ax,$$

$$4x^2-4ax+a^2=3a^2,$$

$$2x-a=\pm\sqrt{3}.a,$$

$$\therefore\ x=\frac{1}{2}(1\pm\sqrt{3}).a.$$

Also $a-2x=\mp x\sqrt{3}.a$, $\therefore$ from (1), $x^2-y^2=\dfrac{\mp 3\sqrt{3}.a^3}{27a}=\mp\dfrac{\sqrt{3}}{9}a^2$,

$$\therefore\ y^2=\left\{\frac{(1\pm\sqrt{3})^2}{4}\pm\frac{\sqrt{3}}{9}\right\}a^2,$$

$$=\left(1\pm\frac{11}{18}\sqrt{3}\right)a^2,\quad \therefore\ y=\sqrt{1\pm\frac{11}{18}\sqrt{3}}.a.$$

Ex. 49. If $\left.\begin{aligned}(b+x)x+ay&=(x+y)^2-2\sqrt{xy}.\sqrt{a-x}.\sqrt{b-y},\\ \sqrt{y}-\sqrt{a-x}&=\sqrt{\frac{nx}{y}},\end{aligned}\right\}$ find x and y.

From 1st, $bx+ay=2xy+y^2-2\sqrt{xy}.\sqrt{a-x}.\sqrt{b-y}$,

$$(a-x)y+(b-y)x+2\sqrt{xy}.\sqrt{a-x}.\sqrt{b-y}=y^2,$$

$$\therefore\ \sqrt{(a-x)y}+\sqrt{(b-y)x}=\pm y\ldots\ldots(1).$$

From 2nd, $\sqrt{(a-x)y}=y-\sqrt{nx}$,

$$\therefore\ \sqrt{(b-y)x}=-\sqrt{nx},$$

$$b-y=n,\quad \therefore\ y=b-n\ldots\ldots\ldots\ldots\ldots(2).$$

Also $(a-x)y=y^2-2y\sqrt{nx}+nx$,

$$ay-y^2=(y+n)x-2y\sqrt{nx},$$

$$=bx-2y\sqrt{nx},\ \text{from (2)},$$

$$\therefore\ x-\frac{2y\sqrt{n}}{b}\sqrt{x}+\left(\frac{y\sqrt{n}}{b}\right)^2=\frac{ny^2}{b^2}+\frac{y}{b}(a-y),$$

$$\therefore\ \sqrt{x}=\frac{y}{b}\left\{\sqrt{n}\pm\sqrt{n+\frac{b}{y}(a-y)}\right\},$$

$$\text{or } \sqrt{x}=\frac{b-n}{b}\left\{\sqrt{n}\pm\sqrt{\frac{ab}{b-n}-(b-n)}\right\},$$

$$\therefore\ x=\left(\frac{b-n}{b}\right)^2\left\{\sqrt{n}\pm\sqrt{\frac{ab}{b-n}-(b-n)}\right\}^2.$$

Ex. 50. If $\left.\begin{aligned} x^2+y^2&=x+\frac{a}{2},\\ x^3(x-2)+6y^2x(x-1)+y^4&=\frac{b}{2}-x,\end{aligned}\right\}$ find x and y.

From 2nd, $x^4-2x^3+6y^2(x^2-x)+y^4=\frac{b}{2}-x,$

$$(x^2-x)^2+6y^2(x^2-x)+y^4=\frac{b}{2}+x^2-x.$$

From 1st, $x^2-x=\frac{a}{2}-y^2,$

$$\therefore\ \left(\frac{a}{2}-y^2\right)^2+6y^2\left(\frac{a}{2}-y^2\right)+y^4=\frac{b}{2}+\frac{a}{2}-y^2,$$

$$\frac{a^2}{4}-ay^2+y^4+3ay^2-6y^4+y^4=\frac{b}{2}+\frac{a}{2}-y^2,$$

$$4y^4-(2a+1)y^2=\frac{a^2}{4}-\frac{a}{2}-\frac{b}{2},$$

$$16y^4-(2a+1)4y^2+\left(\frac{2a+1}{2}\right)^2=a^2-2a-2b+\frac{4a^2+4a+1}{4},$$

$$=\frac{8a^2-4a-8b+1}{4},$$

$$4y^2=\frac{1}{2}\{2a+1\pm\sqrt{8a^2-4a-8b+1}\},$$

$$\therefore\ y=\pm\frac{1}{4}\{4a+2\pm2\sqrt{8a^2-4a-8b+1}\}^{\frac{1}{2}}.$$

Also $x^2-x=\frac{a}{2}-\frac{1}{8}\{2a+1\pm\sqrt{p}\}$, writing p for $8a^2-4a-8b+1$,

$$=\frac{a}{4}-\frac{1}{8}\mp\frac{1}{8}\sqrt{p},$$

$$x^2-x+\frac{1}{4}=\frac{a}{4}+\frac{1}{8}\mp\frac{1}{8}\sqrt{p},$$

$$=\frac{1}{16}\{4a+2\mp2\sqrt{p}\},$$

$$\therefore\ x=\frac{1}{2}\pm\frac{1}{4}\sqrt{\{4a+2\mp2\sqrt{p}\}},$$

$$=\frac{1}{2}\pm\frac{1}{4}\{4a+2\mp2\sqrt{8a^2-4a-8b+1}\}^{\frac{1}{2}}.$$

Ex. 51. If $\left.\begin{array}{l} y\sqrt{(a-x)(x-b)}+x\sqrt{(a-y)(b-y)} \\ \quad =2\{b\sqrt{(a-x)(a-y)}+a\sqrt{(x-b)(b-y)}\}, \\ xy=4ab, \end{array}\right\}$ find x and y.

Square both sides of 1st equation and put $4ab$ for xy, then

$$y^2(ax+bx-ab)+x^2(ab-ay-by)=4\{b^2(xy-ay-ax)+a^2(bx-xy+by)\},$$

$$(x^2-y^2)ab-(a+b)xy(x-y)=4xy(b^2-a^2)-4ab(b-a)(x+y),$$

$$ab(x-y)\{x+y-4(a+b)\}=4ab(a-b)\{x+y-4(a+b)\},$$

$$\therefore\ x-y=4(a-b),\qquad \text{or } x+y=4(a+b),$$

$$\left.\begin{array}{l} x^2-2xy+y^2=16(a-b)^2, \\ 4xy \quad =16ab, \end{array}\right\} \qquad \left.\begin{array}{l} x^2+2xy+y^2=16(a+b)^2, \\ 4xy \quad =16ab, \end{array}\right\}$$

$$\left.\begin{array}{l} x+y=4\sqrt{a^2-ab+b^2}, \\ x-y=4(a-b), \end{array}\right\} \qquad \left.\begin{array}{l} x-y=4\sqrt{a^2+ab+b^2}, \\ x+y=4(a+b), \end{array}\right\}$$

$$\therefore\ \left.\begin{array}{l} x=2(a-b+\sqrt{a^2-ab+b^2}) \\ y=2(\sqrt{a^2-ab+b^2}-a-b) \end{array}\right\}. \qquad \therefore\ \left.\begin{array}{l} x=2(a+b+\sqrt{a^2+ab+b^2}) \\ y=2(a+b-\sqrt{a^2+ab+b^2}) \end{array}\right\}.$$

Ex. 52. If $\left.\begin{array}{r} x^ma^n+y^nb^m=2(ax)^{\frac{m}{2}}.(by)^{\frac{n}{2}}\ldots\ldots(1) \\ xy=ab\ldots\ldots\ldots\ldots\ldots(2) \end{array}\right\}$ find x and y.

From (2), $y^n=a^nb^nx^{-n}$,

$\therefore$ from (1), $a^nx^m+a^nb^{m+n}x^{-n}=2a^{\frac{m}{2}}b^{\frac{n}{2}}x^{\frac{m}{2}}a^{\frac{n}{2}}b^{\frac{n}{2}}.x^{-\frac{n}{2}}=2a^{\frac{m+n}{2}}b^nx^{\frac{m-n}{2}}$,

$$\therefore\ x^{m+n}+b^{m+n}=2a^{\frac{m-n}{2}}b^nx^{\frac{m+n}{2}},$$

$$x^{m+n}-2a^{\frac{m-n}{2}}b^n.x^{\frac{m+n}{2}}+a^{m-n}b^{2n}=a^{m-n}b^{2n}-b^{m+n},$$

$$\therefore\ x^{\frac{m+n}{2}}=a^{\frac{m-n}{2}}b^n\pm\{a^{m-n}b^{2n}-b^{m+n}\}^{\frac{1}{2}},$$

$$=b^n\{a^{\frac{m-n}{2}}\pm(a^{m-n}-b^{m-n})^{\frac{1}{2}}\},$$

$$\therefore\ x = b^{\frac{2n}{m+n}}\{a^{\frac{m-n}{2}} \pm (a^{m-n} - b^{m-n})^{\frac{1}{2}}\}^{\frac{2}{m+n}}.$$

And $y = abx^{-1}$,

$$= ab^{\frac{m-n}{m+n}}.\{a^{\frac{m-n}{2}} \pm (a^{m-n} - b^{m-n})^{\frac{1}{2}}\}^{-\frac{2}{m+n}}.$$

Ex. 56. If $\left.\begin{array}{l}(a+b).\dfrac{y\sqrt{1-x^2}-x\sqrt{1-y^2}}{xy+\sqrt{1-x^2}.\sqrt{1-y^2}} = (a-b).\dfrac{y\sqrt{1-x^2}+x\sqrt{1-y^2}}{xy-\sqrt{1-x^2}.\sqrt{1-y^2}}, \\ a(2y^2-1)+b(2x^2-1)=c, \text{ find } x \text{ and } y.\end{array}\right\}$

For $\dfrac{a+b}{a-b}$ write n, then

$$\frac{ny\sqrt{1-x^2}-nx\sqrt{1-y^2}}{y\sqrt{1-x^2}+x\sqrt{1-y^2}} = \frac{xy+\sqrt{1-x^2}.\sqrt{1-y^2}}{xy-\sqrt{1-x^2}.\sqrt{1-y^2}},$$

$$\frac{(n+1)y\sqrt{1-x^2}-(n-1)x\sqrt{1-y^2}}{(n-1)y\sqrt{1-x^2}-(n+1)x\sqrt{1-y^2}} = \frac{xy}{\sqrt{1-x^2}.\sqrt{1-y^2}}, \text{ Art. 195,}$$

$$(n+1)y(1-x^2)\sqrt{1-y^2}-(n-1)x(1-y^2)\sqrt{1-x^2} = (n-1)xy^2\sqrt{1-x^2} \\ -(n+1)x^2y\sqrt{1-y^2},$$

$$(n+1)y\sqrt{1-y^2}-(n-1)x\sqrt{1-x^2} = 0,$$

$$\frac{n+1}{n-1}.\frac{y}{x} = \sqrt{\frac{1-x^2}{1-y^2}},$$

$$x\sqrt{1-x^2} = \frac{n+1}{n-1}.y\sqrt{1-y^2},$$

$$x^4-x^2+\frac{1}{4} = \frac{1}{4}+\frac{a^2y^2}{b^2}(y^2-1), \quad \because\ \frac{n+1}{n-1} = \frac{a}{b},$$

$$x^2-\frac{1}{2} = \pm\sqrt{\frac{1}{4}+\frac{a^2y^2}{b^2}(y^2-1)},$$

$$2x^2-1 = \pm\sqrt{1+\frac{4a^2y^2}{b^2}(y^2-1)},$$

$$\therefore\ b(2x^2-1) = \pm\sqrt{b^2+4a^2y^2(y^2-1)}.$$

$\therefore$ From 2nd, $a+c-2ay^2 = \pm\sqrt{b^2+4a^2y^2(y^2-1)}$,

$$(a+c)^2-4a(a+c)y^2 = b^2-4a^2y^2,$$

$$4acy^2 = (a+c)^2-b^2,$$

$$\therefore\ y=\frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}.$$

Also $\because$ x and y are similarly involved in the equations, so that if x be put for y and y for x, the equations are not altered, provided a be put for b, and b for a, $\therefore$ we can say at once from the value of y, that

$$x=\frac{1}{2}\sqrt{\frac{(b+c)^2-a^2}{bc}}.$$

Ex. 57. If $\left.\begin{aligned}&\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}(x^{\frac{2}{3}}-1)+\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(2x^{\frac{2}{3}}-1)=\frac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(x^{\frac{1}{3}}+y^{\frac{1}{3}})+\frac{3y}{x^{\frac{1}{3}}}+2,\\ &\frac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}}-\frac{2x^{\frac{2}{3}}}{y}-\frac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}}=\frac{133}{36}\cdot\frac{1}{y^{\frac{2}{3}}}-\frac{2}{x^{\frac{1}{3}}}-\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}},\end{aligned}\right\}$ find x and y.

Write x for $x^{\frac{1}{3}}$, and y for $y^{\frac{1}{3}}$, then

from 1st, $\frac{x}{y}(x^2-1)+\frac{y}{x}(2x^2-1)=\frac{4y}{x}(x+y)+\frac{3y^3}{x}+2,$

$$x^4-x^2+2x^2y^2-y^2=4y^2(x+y)+3y^4+2xy,$$

$$(x^2+y^2)^2=(x+y)^2+4y^2(x+y)+4y^4,$$

$$x^2+y^2=x+y+2y^2,$$

$$x^2-y^2=x+y,\quad \therefore\ x-y=1\ \ldots\ldots\ldots\ (1).$$

From 2nd, $\frac{x^4}{y^4}-\frac{2x^2}{y^3}-\frac{2x}{y}=\frac{133}{36}\cdot\frac{1}{y^2}-\frac{2}{x}-\frac{y^2}{x^2},$

$$\left(\frac{x^2}{y^2}-\frac{1}{y}\right)^2-\frac{2y}{x}\left(\frac{x^2}{y^2}-\frac{1}{y}\right)+\frac{y^2}{x^2}=\frac{133}{36}\cdot\frac{1}{y^2}+\frac{1}{y^2}=\frac{169}{36}\cdot\frac{1}{y^2},$$

$$\frac{x^2}{y^2}-\frac{1}{y}-\frac{y}{x}=\frac{13}{6}\cdot\frac{1}{y},$$

$$\frac{x^2}{y^2}-\frac{y}{x}=\frac{19}{6}\cdot\frac{1}{y},\ \text{ or } x^3-y^3=\frac{19}{6}xy,$$

$\therefore$ dividing by (1), $x^2+xy+y^2=\frac{19}{6}xy,$

$$\therefore\ (x-y)^2 \text{ or } 1=\frac{19}{6}xy-3xy=\frac{xy}{6},$$

$$\therefore\ xy=6, \text{ or } x(x-1)=6,\ x^2-x+\frac{1}{4}=\frac{25}{4},$$

$$\therefore\ x=3, \text{ or } -2, \text{ and } y=2, \text{ or } -3.$$

But calling to mind that x and y were written for $x^{\frac{1}{3}}$ and $y^{\frac{1}{3}}$, the values required are

$$\begin{cases} x=27, & \text{or } -8, \\ y=8, & \text{or } -27. \end{cases}$$

Ex. 59. If $\left.\begin{aligned} x^4&=mx+ny, \\ y^4&=my+nx, \end{aligned}\right\}$ find x and y.

$$\frac{x^4}{y^4}=\frac{mx+ny}{my+nx}=\frac{m\frac{x}{y}+n}{m+n\frac{x}{y}}, \text{ write } t \text{ for } \frac{x}{y},$$

$$t^4=\frac{mt+n}{m+nt}, \text{ or } mt^4+nt^5=mt+n,$$

$$mt(t^3-1)+n(t^5-1)=0,$$

$$mt(t^2+t+1)+n(t^4+t^3+t^2+t+1)=0, \text{ and } t-1=0, \text{ or } t=1 \ldots (1)$$

$$nt^4+(m+n)t^3+(m+n)t^2+(m+n)t+n=0,$$

$$n\left(t^2+\frac{1}{t^2}\right)+(m+n)\left(t+\frac{1}{t}\right)+(m+n)=0,$$

$$\left(t+\frac{1}{t}\right)^2+\frac{m+n}{n}\left(t+\frac{1}{t}\right)+\left(\frac{m+n}{2n}\right)^2=\frac{(m+n)^2}{4n^2}-\frac{m+n}{n}+2,$$

$$=\frac{m^2-2mn+5n^2}{4n^2},$$

$$t+\frac{1}{t}=\frac{-(m+n)\pm\sqrt{m^2-2mn+5n^2}}{2n}=a, \text{ suppose,}$$

$$t^2-at=-1,\ \therefore\ t=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-1} \ldots\ldots (2).$$

From (1), $x=y=\sqrt[3]{m+n}$, substituting in the 1st equation.

From (2), and 2nd equation,

$$y=\sqrt[3]{m+nt}=\sqrt[3]{m+\frac{n}{2}(a\pm\sqrt{a^2-4})},$$

$$\text{and } x=\sqrt[3]{m+\frac{n}{t}}=\sqrt[3]{m+\frac{n}{\frac{1}{2}(a\pm\sqrt{a^2-4})}},$$

$$=\sqrt[3]{m+\frac{n}{2}(a\mp\sqrt{a^2-4})}.$$

Ex. 64. If
$$\left.\begin{array}{l} 30\sqrt{\dfrac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{4}{3}}y^{\frac{4}{3}}}}+40\sqrt{\dfrac{x^{\frac{4}{3}}y^{\frac{4}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}=241, \\ \left\{1+\left(\dfrac{y}{x}\right)^{\frac{2}{3}}\right\}\cdot\left\{3x^{\frac{4}{3}}y^{\frac{2}{3}}+\dfrac{91}{216}\sqrt{x^2+x^{\frac{4}{3}}y^{\frac{2}{3}}}\right\}=\left(\dfrac{5}{6}\right)^3-x^2-y^2, \end{array}\right\}\text{ find } x \text{ and } y.$$

From 2nd, $3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}+\dfrac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=\dfrac{125}{216}-x^2-y^2,$

$$x^2+3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}+y^2+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=\frac{125}{216},$$

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^3+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=\frac{125}{216},$$

$z^2+\dfrac{91}{216}z=\dfrac{125}{216}$, writing z for $(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}$,

$$z^2+\frac{91}{216}z+\left(\frac{91}{432}\right)^2=\frac{8281}{(432)^2}+\frac{125}{216}=\frac{116281}{(432)^2},$$

$$\therefore\ z=\frac{-91\pm341}{432}=\frac{125}{216},\text{ or } -1;$$

$$\therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{25}{36}\text{..........................}(1).$$

From 1st, $30(x^{\frac{2}{3}}+y^{\frac{2}{3}})+40xy=241\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}\times x^{\frac{2}{3}}y^{\frac{2}{3}},$

$$30\times\frac{25}{36}+40xy=241\times\frac{5}{6}\times x^{\frac{2}{3}}y^{\frac{2}{3}},$$

$$25+48xy=241x^{\frac{2}{3}}y^{\frac{2}{3}},$$

$$25-x^{\frac{2}{3}}y^{\frac{2}{3}}=48x^{\frac{2}{3}}y^{\frac{2}{3}}(5-x^{\frac{1}{3}}y^{\frac{1}{3}}),$$

$$\therefore\ 5-x^{\frac{1}{3}}y^{\frac{1}{3}}=0,\text{ and } 5+x^{\frac{1}{3}}y^{\frac{1}{3}}=48x^{\frac{2}{3}}y^{\frac{2}{3}},$$

$$x^{\frac{1}{3}}y^{\frac{1}{3}}=5,$$

From (1) $\left.\begin{aligned} x^{\frac{2}{3}}+y^{\frac{2}{3}}&=\frac{25}{36}, \\ \text{and } 2x^{\frac{1}{3}}y^{\frac{1}{3}}&=10, \end{aligned}\right\}$

$$\therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=\frac{\sqrt{385}}{6},$$

and $x^{\frac{1}{3}}-y^{\frac{1}{3}}$ = an impossible quantity.

---

or $x^{\frac{2}{3}}y^{\frac{2}{3}}-\frac{1}{48}x^{\frac{1}{3}}y^{\frac{1}{3}}+\left(\frac{1}{96}\right)^2=\frac{1}{(96)^2}+\frac{5}{48}=\frac{961}{(96)^2}$,

$$x^{\frac{1}{3}}y^{\frac{1}{3}}=\frac{1\pm31}{96}=\frac{1}{3}, \text{ or } -\frac{5}{16},$$

From (1) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{25}{36}$,

and $2x^{\frac{1}{3}}y^{\frac{1}{3}}=\frac{2}{3}$,

$$\left.\begin{aligned} \therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}&=\pm\frac{7}{6}, \\ x^{\frac{1}{3}}-y^{\frac{1}{3}}&=\pm\frac{1}{6}, \end{aligned}\right\}$$

$$\therefore\ 2x^{\frac{1}{3}}=\pm\frac{4}{3}, \text{ or } \pm1, \ \therefore\ x=\pm\frac{8}{27}, \text{ or } \pm\frac{1}{8},$$

$$2y^{\frac{1}{3}}=\pm1, \text{ or } \pm\frac{4}{3}, \ \therefore\ y=\pm\frac{1}{8}, \text{ or } \pm\frac{8}{27}.$$

Ex. 65. If

$$\left.\begin{aligned} (x^4+2bx^2y+a^2y^2)(y^4+2bxy^2+a^2x^2)&=4(a^2-b^2)(b+c)^2(xy)^2, \ldots\ldots(1) \\ x^3+y^3&=2cxy, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \end{aligned}\right\}$$

find x and y.

Divide (1) by $(xy)^2$, then

$$\left(x^2+2by+\frac{a^2y^2}{x^2}\right)\left(y^2+2bx+\frac{a^2x^2}{y^2}\right)=4(a^2-b^2)(b+c)^2,$$

$$x^2y^2+2bx^3+\frac{a^2x^4}{y^2}+2by^3+4b^2xy+2a^2b\frac{x^2}{y}+\frac{a^2y^4}{x^2}+2a^2b\frac{y^2}{x}+a^4$$
$$=4(a^2-b^2)(b+c)^2,$$

$$x^2y^2+2b(x^3+y^3)+4b^2xy+a^2\left(\frac{x^4}{y^2}+\frac{y^4}{x^2}\right)+2a^2b\left(\frac{x^2}{y}+\frac{y^2}{x}\right)+a^4$$
$$=4(a^2-b^2)(b+c)^2,$$

$$x^2y^2+4bcxy+4b^2xy+a^2\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^2-2a^2xy+2a^2b\,.\,2c+a^4$$
$$=4(a^2-b^2)(b+c)^2,$$

$$x^2y^2+(4bc+4b^2-2a^2)xy+4a^2c^2+4a^2bc+a^4=4a^2b^2+8a^2bc+4a^2c^2$$
$$-4b^4-8b^3c-4b^2c^2,$$

$$x^2y^2+(4bc+4b^2-2a^2)xy+4b^2c^2+4b^4+a^4-4a^2b^2-4a^2bc+8b^3c=0,$$

$$x^2y^2+(4bc+4b^2-2a^2)xy+(2bc+2b^2-a^2)^2=0,$$

$$xy+2bc+2b^2-a^2=0,$$

$$\therefore\ xy=a^2-2bc-2b^2=p^2, \text{ suppose.}$$

Then from (2), $x^3+\dfrac{p^6}{x^3}=2cp^2$,

$$x^6-2cp^2x^3+(cp^2)^2=c^2p^4-p^6,$$

$$x^3=cp^2\pm\sqrt{c^2p^4-p^6},$$

$$=p^2\{c\pm\sqrt{c^2-p^2}\},$$

$\therefore\ x=\sqrt[3]{p^2}.\sqrt[3]{c\pm\sqrt{c^2-p^2}}$, where $p^2=a^2-2bc-2b^2$.

And $y=\dfrac{p^2}{x}=\dfrac{\sqrt{p^4}}{\sqrt[3]{c\pm\sqrt{c^2-p^2}}}=\sqrt[3]{p^2}.\sqrt[3]{c\mp\sqrt{c^2-p^2}}$.

Ex. 66. If $\left.\begin{array}{l} a^2-x^2=3xy,\ldots\ldots\ldots\ldots\ldots(1)\\ (\sqrt{y}-\sqrt{x})(a-x)=3\sqrt{x}(x+y)\ldots\ldots(2)\end{array}\right\}$ find x and y.

From (1), $(a+x)(a-x)=3\sqrt{x}.y\sqrt{x}$,

from (2), $(\sqrt{y}-\sqrt{x})(a-x)=3\sqrt{x}(x+y)$,

$$\therefore\ \frac{\sqrt{y}-\sqrt{x}}{a+x}=\frac{x+y}{y\sqrt{x}},$$

$$\therefore\ y\sqrt{xy}-xy=(a+x)(x+y)\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

But $y=\dfrac{a^2-x^2}{3x}$, $\therefore\ xy=\dfrac{a^2-x^2}{3}$, and $x+y=\dfrac{a^2+2x^2}{3x}$,

$\therefore$ substituting in (3), $\dfrac{a^2-x^2}{3x}\sqrt{\dfrac{a^2-x^2}{3}}=\dfrac{a^2-x^2}{3}+(a+x).\dfrac{a^2+2x^2}{3x}$,

$$(a-x)\sqrt{\frac{a^2-x^2}{3}}=(a-x)x+a^2+2x^2,$$

$$=a^2+ax+x^2,$$

$$\therefore\ a^4-x^4-2ax(a^2-x^2)=3(a^2+ax+x^2)^2,$$

$$a^4-x^4-2a^3x+2ax^3=3(a^4+x^4+2a^2x^2+a^2x^2+2a^3x+2ax^3),$$

$$4x^4+4ax^3+9a^2x^2+8a^3x+2a^4=0,$$

$$4x^2(x^2+2a^2)+a^2(x^2+2a^2)+4ax(x^2+2a^2)=0,$$

$$\therefore\ x^2+2a^2=0,$$

$$x=\pm a\sqrt{-2}.$$

$$\text{And } y=\pm\frac{a^2+2a^2}{3a\sqrt{-2}}=\pm\frac{a}{\sqrt{-2}}.$$

$$\text{and } 4x^2+a^2+4ax=0,$$

$$\text{or } 2x+a=0,\ \therefore\ x=-\frac{a}{2}.$$

$$\text{And } y=\frac{a^2-x^2}{3x}=-\frac{a}{2}.$$

Ex. 67. If $\left.\begin{array}{l}(1-x^2)^2(1+y^2)-(1+x^2)^2(1-y^2)=4x^2\sqrt{1+y^4}\ldots(1)\\ 4xy=\sqrt{2}(1-x^2)(1-y^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{array}\right\},$

find x and y.

From (1), $(1-2x^2+x^4)(1+y^2)-(1+2x^2+x^4)(1-y^2)=4x^2\sqrt{1+y^4}$,

$$2y^2-4x^2+2x^4y^2=4x^2\sqrt{1+y^4},$$

$$y^2(1+x^4)=2x^2(1+\sqrt{1+y^4})\ldots\ldots\ldots\ldots(3).$$

From (2), $$(1-x^2)^2=\frac{8x^2y^2}{(1-y^2)^2},$$

$$\therefore\ 1+x^4=\frac{8x^2y^2}{(1-y^2)^2}+2x^2,$$

$$=\frac{2x^2(1+y^2)^2}{(1-y^2)^2}.$$

Substituting in (3), $$\frac{y^2(1+y^2)^2}{(1-y^2)^2}=1+\sqrt{1+y^4},$$

$$1+y^4=\frac{y^4(1+y^2)^4}{(1-y^2)^4}-\frac{2y^2(1+y^2)^2}{(1-y^2)^2}+1,$$

$$y^2(1-y^2)^4=y^2(1+y^2)^4-2(1-y^4)^2,$$

$$4y^4+4y^8=(1-y^4)^2=1-2y^4+y^8,$$

$$3y^8+6y^4=1,$$

$$y^8+2y^4+1=\frac{4}{3},$$

$$y^4+1=\frac{2}{\sqrt{3}},$$

$$y^4=\frac{2-\sqrt{3}}{\sqrt{3}}=\frac{4-2\sqrt{3}}{2\sqrt{3}},$$

$$\therefore\ y^2=\pm\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}},\ \text{ and } y=\pm\sqrt{\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}}.$$

Again, from (2) $x^2+\frac{2\sqrt{2}.y}{1-y^2}.x=1$,

$$\therefore\ x^2+\frac{2\sqrt{2}.y}{1-y^2}.x+\frac{2y^2}{(1-y^2)^2}=1+\frac{2y^2}{(1-y^2)^2}=\frac{1+y^4}{(1-y^2)^2},$$

$$\therefore\ x=\frac{-\sqrt{2}.y\pm\sqrt{1+y^4}}{1-y^2},$$

$$=\frac{\mp\frac{\sqrt{2\sqrt{3}-2}}{\sqrt[4]{2\sqrt{3}}}\pm\frac{2}{\sqrt{2\sqrt{3}}}}{\frac{\sqrt{2\sqrt{3}}\mp\sqrt{3}\pm1}{\sqrt{2\sqrt{3}}}},$$

$$=\frac{\mp\sqrt{2\sqrt{3}-2}.\sqrt[4]{2\sqrt{3}}\pm2}{\sqrt{2\sqrt{3}}\mp(\sqrt{3}-1)}.$$

Ex. 13. Given $x+y+z=\frac{14}{3}x=\frac{7}{2}y$, find the value of $\frac{x+y+z}{z}$.

$$\frac{x+y+z}{14}=\frac{x}{3}=\frac{y}{4},$$

by Art. 195*, $\frac{x+y+z}{14}=\frac{(x+y+z)-x-y}{14-3-4}$,

$$=\frac{z}{7},$$

$$\therefore\ \frac{x+y+z}{z}=\frac{14}{7}=2.$$

Ex. 14. Given $x-y=7z$, and $x-z=4y$, find the value of $\frac{y-z}{x}$.

$$7z+y=x=4y+z,$$

$$\therefore\ 6z=3y,$$

$$\text{or } 2z=y,\ \therefore\ y-z=z,$$

$$\therefore\ \frac{y-z}{x}=\frac{z}{x}=\frac{z}{7z+y}=\frac{z}{9z}=\frac{1}{9}.$$

Ex. 15. Given $x^2+y^2=123z$, and $x^2-y^2=27z$, find the value of $\frac{xy}{z}$.

$$x^2+y^2=123z,$$
$$x^2-y^2=27z,$$
$$\therefore\ 2x^2=150z,$$
$$\therefore\ x^2=75z.$$
$$\text{Also } 2y^2=96z,$$
$$\therefore\ y^2=48z,$$
$$\therefore\ x^2y^2=75\times 48 . z^2,$$
$$\frac{x^2y^2}{z^2}=75\times 48=25\times 144,$$
$$\therefore\ \frac{xy}{z}=5\times 12=60.$$

Ex. 16. Given $\left.\begin{array}{l} ab-\frac{1}{2}(a+b)(p+q)+pq=0, \ldots\ldots (1) \\ \text{and } cd-\frac{1}{2}(c+d)(p+q)+pq=0, \ldots\ldots (2) \end{array}\right\}$

shew that $\left(\frac{p-q}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}$.

Subtracting (2) from (1), $ab-cd-\frac{1}{2}(a+b-c-d)(p+q)=0$,

$$\therefore\ p+q=\frac{2(ab-cd)}{a+b-c-d}.$$

Again, $\frac{a+b}{c+d}=\frac{ab+pq}{cd+pq}$, from (1) and (2),

$$\therefore\ (a+b-c-d)pq=ab(c+d)-cd(a+b),$$

$$\therefore\ (a+b-c-d)^2(\overline{p+q}|^2-4pq)=4(ab-cd)^2-4(a+b-c-d)\{ab(c+d)-cd(a+b)\},$$

$$\begin{aligned}(a+b-c-d)^2\left(\frac{p-q}{2}\right)^2 &=(ab-cd)^2-(a+b-c-d)(abc+abd-acd-bcd),\\ &=(ab-cd)^2-(a-d+b-c)\{(a-d)bc+(b-c)ad\},\\ &=a^2b^2+c^2d^2-2abcd-\big[bc(a-d)^2+bc(a-d)(b-c)\\ &\qquad +ad(a-d)(b-c)+ad(b-c)^2\big],\\ &=a^2b^2+c^2d^2+2abcd-a^2bc-bcd^2-ab^2d-ac^2d\\ &\qquad -(a-d)(b-c)(ad+bc),\\ &=a^2b(b-c)-cd^2(b-c)+(acd-abd)(b-c)\\ &\qquad -(a-d)(b-c)(ad+bc),\end{aligned}$$

$$=(b-c)\{a^2b-cd^2+acd-abd-(a-d)(ad+bc)\},$$

$$=(b-c)\{ab(a-d)+cd(a-d)-(ad+bc)(a-d)\},$$

$$=(a-d)(b-c)\{ab+cd-ad-bc\},$$

$$=(a-d)(b-c)\{a(b-d)-c(b-d)\},$$

$$=(a-d)(b-c)(a-c)(b-d),$$

$$\therefore \left(\frac{p-q}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}.$$

Ex. 17. If the same value of x satisfies both the equations $ax^2+bx+c=0$, and $a'x^2+b'x+c'=0$, what is the relation subsisting between a, b, c, a', b', c'?

Since the two given equations hold for the same value of x,

$$\left.\begin{aligned}\therefore\ aa'x^2+a'bx+a'c=0,\\ \text{and } aa'x^2+ab'x+ac'=0,\end{aligned}\right\}\text{ hold for the same value of } x,$$

$$\therefore\ (ab'-a'b)x=a'c-ac',$$

$$\therefore\ x=\frac{a'c-ac'}{ab'-a'b}.$$

$$\left.\begin{aligned}\text{Again }\ ac'x^2+bc'x+cc'=0,\\ \text{and }\ a'cx^2+b'cx+cc'=0,\end{aligned}\right\}$$

$$\therefore\ (a'c-ac')x^2=(bc'-b'c)x,$$

$$\therefore\ x=\frac{bc'-b'c}{a'c-ac'}.$$

$$\therefore\ \frac{a'c-ac'}{ab'-a'b}=\frac{bc'-b'c}{a'c-ac'},$$

$$\therefore\ (ab'-a'b)(bc'-b'c)=(a'c-ac')^2,\ \text{or}\ (ac'-a'c)^2.$$

Ex. 18. Find the relation subsisting between the coefficients in the equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, $a_3x+b_3y=c_3$, that they may be satisfied by the same values of x and y.

$$\left.\begin{aligned}a_1x+b_1y=c_1\ldots\ldots(1)\\ a_2x+b_2y=c_2\ldots\ldots(2)\\ a_3x+b_3y=c_3\ldots\ldots(3)\end{aligned}\right\}\text{ hold together.}$$

From (1) and (2), $\left.\begin{aligned} a_1b_2x+b_1b_2y&=b_2c_1 \\ a_2b_1x+b_1b_2y&=b_1c_2 \end{aligned}\right\}$

$$\therefore\ (a_1b_2-a_2b_1)x=b_2c_1-b_1c_2,$$

$$\therefore\ x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}.$$

Again from (1) and (2), $\left.\begin{aligned} a_1a_2x+a_2b_1y&=a_2c_1 \\ a_1a_2x+a_1b_2y&=a_1c_2 \end{aligned}\right\}$

$$\therefore\ (a_1b_2-a_2b_1)y=a_1c_2-a_2c_1,$$

$$\therefore\ y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.$$

Substituting the values of x and y thus found in (3), we have

$$\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}.a_3+\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.b_3=c_3,$$

$$(b_2c_1-b_1c_2)a_3+(a_1c_2-a_2c_1)b_3=(a_1b_2-a_2b_1)c_3,$$

$$\text{or}\ \ a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0.$$

Ex. 19. Find the relations subsisting between the coefficients, when the equations

$$ax+by+cz=a'x+b'y+c'z=a''x+b''y+c''z=1,$$

are equivalent to no more than two distinct equations. And shew that in this case the values of x, y, z, assume the form $0\div 0$.

$$\left.\begin{aligned} ax+by+cz&=1\ldots\ldots\ldots(1) \\ a'x+b'y+c'z&=1\ldots\ldots\ldots(2) \\ a''x+b''y+c''z&=1\ldots\ldots\ldots(3) \end{aligned}\right\}.$$

From (1) and (2), $(a-a')\dfrac{x}{z}+(b-b')\dfrac{y}{z}+c-c'=0,$

... (2) and (3), $(a'-a'')\dfrac{x}{z}+(b'-b'')\dfrac{y}{z}+c'-c''=0.$

But as x, y, z, by supposition, cannot be determined from the *given* equations, since the *three* equations are equivalent to no more than *two*, therefore the values of $\dfrac{x}{z}$, and $\dfrac{y}{z}$, are arbitrary. Hence, representing the two equations above by

$$AX+BY+C=0,\ \text{and}\ A'X+B'Y+C'=0,$$

$$AB'X+BB'Y+B'C=0,\ \text{and}\ A'BX+BB\,Y+BC'=0,$$

$\therefore\ AB'X+B'C=A'BX+BC'$, an *identical* equation,

$$\therefore\ AB' = A'B, \text{ and } B'C = BC',$$

$$\text{or } \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'};\ \text{or } \frac{a-a'}{a'-a''} = \frac{b-b'}{b'-b''} = \frac{c-c'}{c'-c''}.$$

Also, obtaining the values of x, y, z from the given equations, by the usual method, the relations subsisting between the coefficients, as here found, will make each equal to $0 \div 0$.

Ex. 21. Find the value of $ax^2 + by^2 + cz^2$, when $ax^3 = by^3 = cz^3$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{d}$.

$$\text{Here } x\sqrt[3]{a} = y\sqrt[3]{b} = z\sqrt[3]{c},$$

$$\therefore\ \frac{1}{x} = \sqrt[3]{\frac{a}{c}} \cdot \frac{1}{z},$$

$$\frac{1}{y} = \sqrt[3]{\frac{b}{c}} \cdot \frac{1}{z},$$

$$\frac{1}{z} = \sqrt[3]{\frac{c}{c}} \cdot \frac{1}{z}.$$

$$\therefore\ \frac{1}{d} = \frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{\sqrt[3]{c}} \cdot \frac{1}{z},$$

$$\therefore\ z = d.\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{\sqrt[3]{c}}.$$

$$\text{Similarly, } y = d.\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{\sqrt[3]{b}},$$

$$\text{and } x = d.\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{\sqrt[3]{a}}.$$

$$\therefore\ ax^2 + by^2 + cz^2 = a^{\frac{1}{3}}(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^2 d^2 + b^{\frac{1}{3}}(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^2 d^2 + c^{\frac{1}{3}}(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^2 d^2,$$

$$= (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 . d^2.$$

Another Method. Apply the formula of Art. 195*.

Ex. 22. Eliminate a, b, c, from the equations

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m + \left(\frac{z}{c}\right)^m = 1 = \left(\frac{a}{d}\right)^n + \left(\frac{b}{d}\right)^n + \left(\frac{c}{d}\right)^n, \text{ and } \frac{x^m}{a^{m+n}} = \frac{y^m}{b^{m+n}} = \frac{z^m}{c^{m+n}}.$$

Since $\frac{x^m}{a^{m+n}}+\frac{y^m}{b^{m+n}}=\frac{z^m}{c^{m+n}}$,

$$\left(\frac{y}{b}\right)^m=\frac{b^n}{a^n}\left(\frac{x}{a}\right)^m, \text{ and } \left(\frac{z}{c}\right)^m=\frac{c^n}{a^n}\left(\frac{x}{a}\right)^m,$$

$$\therefore \left(\frac{x}{a}\right)^m+\frac{b^n}{a^n}\left(\frac{x}{a}\right)^m+\frac{c^n}{a^n}\left(\frac{x}{a}\right)^m=1,$$

$$\therefore \left(\frac{x}{a}\right)^m=\frac{a^n}{a^n+b^n+c^n}=\frac{a^n}{d^n},$$

$$\therefore \frac{a^{m+n}}{d^n}=x^m, \text{ or } \left(\frac{a}{d}\right)^{m+n}=\left(\frac{x}{d}\right)^m,$$

$$\text{and} \therefore \left(\frac{a}{d}\right)^n=\left(\frac{x}{d}\right)^{\frac{mn}{m+n}}.$$

$$\text{Similarly, } \left(\frac{b}{d}\right)^n=\left(\frac{y}{d}\right)^{\frac{mn}{m+n}},$$

$$\text{and } \left(\frac{c}{d}\right)^n=\left(\frac{z}{d}\right)^{\frac{mn}{m+n}},$$

$$\therefore \left(\frac{x}{d}\right)^{\frac{mn}{m+n}}+\left(\frac{y}{d}\right)^{\frac{mn}{m+n}}+\left(\frac{z}{d}\right)^{\frac{mn}{m+n}}=1,$$

$$\text{or } x^{\frac{mn}{m+n}}+y^{\frac{mn}{m+n}}+z^{\frac{mn}{m+n}}=a^{\frac{mn}{m+n}}.$$

Another Method. Apply the formula of Art. 195*.

PROBLEMS.

PROB. 28. A person, being asked what o'clock it was, answered that it was between 5 and 6, and that the hour and minute hands were together. Required the time of day.

When it was 5 o'clock the hour hand was 5 hours, or 25 of the minute divisions, in advance of the minute hand; $\therefore$ if x be the number of minutes past 5, when the hour and minute hands are together, the minute hand travels over x of the minute divisions, while the hour hand travels over $x-25$. But the minute hand goes uniformly 12 times as fast as the hour hand, and will therefore travel over 12 times the space in the same time. Hence

$$12(x-25)=x,$$

$$11x=300,$$

$$\therefore\ x=\frac{300}{11}=27\text{m. }16\tfrac{4}{11}\text{s.}$$

PROB. 29. Find the time after h o'clock, at which the hour and minute hands of a watch are distant d of the minute divisions from each other.

Let x = the number of minutes past h o'clock,

then the minute hand goes over x of the minute divisions, while the hour hand goes over $x-5h\mp d$, $-$ or $+$ according as the minute hand is in advance or behind the hour hand,

$$\therefore \text{ as before, }\ 12(x-5h\mp d)=x,$$

$$11x=12(5h\mp d),$$

$$x=\frac{12}{11}(5h\pm d).$$

COR. Let $h=5$, and $d=0$, then $x=\frac{12}{11}\times 25=\frac{300}{11}$, which agrees with the preceding problem.

PROB. 35. A hare is 80 of its own leaps before a greyhound, and takes 3 leaps for every 2 taken by the greyhound, but the latter passes over as much ground in one leap as the former does in two. How many leaps will the hare have taken before it is caught?

Let the hare take $3x$ leaps after the greyhound starts. Then the latter takes $2x$ leaps in catching the hare. If each of the hare's leaps passes over a certain measure of ground, each leap of the greyhound passes over *two* of such measures, and his $2x$ leaps over $4x$ of these measures.

$$\therefore\ 80+3x=4x, \text{ by the question;}$$

$$\therefore\ x=80,$$

and $3x=240$, number of leaps required.

PROB. 38. A banker has two kinds of money, silver and gold; and a pieces of silver, or b pieces of gold, make up the same sum s. A person comes, and wishes to be paid the sum s with c pieces of money: how many of each must the banker give him?

Let x be the number of pieces of silver required,

then $c-x=$ .... gold ....

and $\frac{s}{a}=$ the value of *one* of the former,

$\frac{s}{b}=$ .... latter,

and $\frac{sx}{a}+\frac{s(c-x)}{b}=s$, the value of the whole,

$$\frac{x}{a}+\frac{c-x}{b}=1,$$

$$bx+ac-ax=ab,$$

$$(a-b)x=ac-ab, \quad \therefore\ x=a\,.\frac{c-b}{a-b}.$$

$$\text{Also } c-x=c-\frac{ac-ab}{a-b}=b\,.\frac{a-c}{a-b}.$$

PROB. 51. When wax candles are half-a-crown a pound, a composition is invented of such a nature, that a candle made of it will burn two thirds as long as a wax candle one fourth as heavy again will burn. Supposing the two candles give an equally bright light, what must be charged per lb. for the composition that it may be as *cheap* as wax?

Let x be the price required per lb. of the composition, in pence, t the time, in minutes, in which 1 lb. of the wax candle will burn,

then $t+\frac{t}{4}$, or $\frac{5t}{4}=$ time of burning $1\frac{1}{4}$ lbs. of wax,

$\therefore\ \frac{2}{3}\times\frac{5t}{4}=$ time of burning 1 lb. of the composition candle,

$\therefore\ \frac{30}{t}=$ cost of wax per minute, in pence,

$\frac{x}{\frac{2}{3}\times\frac{5t}{4}}=$ cost of composition, ...

$\therefore$ by the question, $\frac{x}{\frac{2}{3}\times\frac{5t}{4}}=\frac{30}{t}$,

$$\therefore\ x=\frac{2}{3}\times\frac{5}{4}\times 30=25=2s.\ 1d.$$

PROB. 54. A and B engaged to reap equal quantities of wheat, and A began half an hour before B. They stopped at 12 o'clock, and rested an hour, observing that just half the whole work was done. B's part was finished at 7 o'clock, and A's at a quarter before 10. Supposing them to have laboured uniformly, find the times at which they commenced.

Let x = the number of hours before noon when A began,

then $x-\frac{1}{2}=$ .... B ......

Now since A and B reaped *equal* quantities in the whole day, and half the whole work was done by noon,

A did as much in x hours before noon, as B did in 6 hours in the afternoon; and again

A did as much in $8\frac{3}{4}$ hours in the afternoon, as B did in $x-\frac{1}{2}$ hours in the forenoon.

If then a represent A's hourly work, and b B's,

$$\left.\begin{array}{r} ax=6b, \text{ ....(1)} \\ \text{and } 8\tfrac{3}{4}\times a=(x-\tfrac{1}{2})b, \text{ ....(2)} \end{array}\right\}$$

$\therefore$ dividing (2) by (1), $\dfrac{8\frac{3}{4}}{x}=\dfrac{x-\frac{1}{2}}{6}$,

$$x^2-\frac{x}{2}=52\tfrac{1}{2}=\frac{105}{2},$$

$$x^2-\frac{x}{2}+\frac{1}{16}=\frac{105}{2}+\frac{1}{16}=\frac{841}{16},$$

$$\therefore\ x=\frac{1}{4}+\frac{29}{4}=\frac{30}{4}=7\tfrac{1}{2}.$$

$\therefore$ A began at $\frac{1}{2}$ past 4; B at 5 o'clock.

PROB. 61. The mail-train upon a railway starts a certain time after a luggage-train from the same terminus, and the time is so adjusted that, before arriving at the other terminus, the trains will exactly escape collision and no more. It happens, however, that from an accident to the engine the speed of the luggage train is suddenly reduced one-half after performing two-thirds of its journey, and a collision takes place a miles from the end of it. The proper speeds of the trains being m and n miles per hour, $(m>n)$ find the length of the railway, and the difference of the times of starting.

Let x = length of the railway, in miles,

y = difference of starting time, in hours,

$\therefore \frac{x}{m}$ = time in which the mail should perform its journey,

and $\frac{x}{n}$ = ... luggage-train ....

$$\therefore \frac{x}{m} = \frac{x}{n} - y \ldots\ldots (1).$$

Again, $\frac{x-a}{m}$ = time by the mail before the collision,

$\frac{2x}{3n}$ = time by luggage-train before the accident to the engine,

and $\left(\frac{x}{3} - a\right) \div \frac{n}{2}$ = time between the accident and the collision,

$$\therefore \frac{x-a}{m} = \frac{2x}{3n} + \frac{2x-6a}{3n} - y, \ldots\ldots (2),$$

and subtracting (2) from (1),

$$\frac{a}{m} = \frac{x}{3n} - \frac{2x}{3n} + \frac{2a}{n},$$

$$\therefore \frac{x}{3n} = \frac{2a}{n} - \frac{a}{m},$$

$$\therefore x = 3\left(2 - \frac{n}{m}\right).a.$$

$$\therefore \text{Also } y = \frac{x}{n} - \frac{x}{m} = \frac{m-n}{mn}.x = 3.\frac{m-n}{mn}\left(2 - \frac{n}{m}\right).a.$$

Prob. 64. A pack of p cards is distributed into n heaps, so that the number of pips on the lowest cards, together with the number of cards laid upon them, is the same number m for each heap, and the number of cards remaining is found to be r; required the number of pips on all the lowest cards.

Let x = the number of pips required; then

mn = number of pips on the lowest cards + number of cards upon them,

$\therefore mn - x$ = number of cards laid upon the lowest;

$\therefore mn - x + n$ = number of cards in all the heaps,

$= p - r,$

$$\therefore x = (m+1)n + r - p.$$

Prob. 65. If a men or b boys can dig m acres in n days, find the number of boys whose assistance will be required to enable $(a-p)$ men to dig $(m+p)$ acres in $(n-p)$ days.

Let x = the number of boys required; then

since a men dig m acres in n days,

$$a \text{ men dig } \frac{m}{n} \ \ldots\ldots \ 1 \text{ day},$$

$$\text{and } 1 \text{ man digs } \frac{m}{na} \ \ldots\ldots \ 1 \text{ day}.$$

$$\text{Similarly, } 1 \text{ boy digs } \frac{m}{nb} \ \ldots\ldots \ 1 \text{ day}.$$

$$\therefore (a-p)\frac{m}{na} = \text{work of } (a-p) \text{ men in } 1 \text{ day},$$

$$\text{and } (n-p)(a-p)\frac{m}{na} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (n-p) \text{ days}.$$

$$\text{Also } (n-p)x.\frac{m}{nb} = \ldots\ldots\ldots\ldots\ldots\ldots x \text{ boys in } (n-p) \text{ days}.$$

$$\therefore (n-p)\frac{m}{nb}x + (n-p)(a-p)\frac{m}{na} = m+p,$$

$$(n-p)\frac{m}{nb}x = m+p-(n-p)(a-p)\frac{m}{na},$$

$$= m+p-m+\frac{mp}{n}+\frac{mp}{na}(n-p),$$

$$= \frac{mp}{n}\left\{1+\frac{n}{m}+\frac{n-p}{a}\right\},$$

$$\therefore x = \frac{pb}{a}\left\{n-p+(m+n)\frac{a}{m}\right\} \div (n-p),$$

$$= \frac{pb}{a}\left(1+\frac{m+n}{n-p}\cdot\frac{a}{m}\right).$$

Prob. 66. Supposing the sum of 51 cards in a common pack to be $10n+a$, (where $a<10$), prove the value of the last card to be $10-a$, the court-cards reckoning for 10, and the others for as much as is the number of pips upon each. Find also the value of n.

Let x = the value of the last card.

The sum of the values of all the cards

$$= 4(1+2+3+\ldots 10)+4\times 3\times 10,$$

$$= 4\times 55+120 = 340,$$

$$\therefore 10n+a+x = 340.$$

But $a<10$, and $x<10$, $\therefore$ $10n>320$, and <340,

$$\therefore\ n>32, \text{ and } <34, \therefore\ n=33.$$

$$\therefore\ 330+a+x=340,$$

$$\therefore\ a+x=10,$$

$$\therefore\ x=10-a.$$

PROB. 67. If a oxen in m weeks eat b acres of grass, and c oxen eat d acres in n weeks, how many oxen will eat e acres in p weeks, supposing the grass to grow uniformly?

Let $x=$ the number of oxen required,

$g=$ the grass upon one acre at first,

$h=$ the weekly growth of one acre,

then a oxen eat $b(g+mh)$ in m weeks,

$\therefore$ 1 ox eats $\frac{b}{a}(g+mh)$,..........

and 1 ox eats $\frac{b}{ma}(g+mh)$ in 1 week............(1).

Similarly, from the 2nd supposition,

1 ox eats $\frac{d}{nc}(g+nh)$ in 1 week, ...(2).

and again, 1 ox eats $\frac{e}{px}(g+ph)$ in 1 week, ...(3).

$\therefore$ we have $\frac{b}{ma}(g+mh)=\frac{d}{nc}(g+nh)=\frac{e}{px}(g+ph)$, to eliminate g and h.

1st. $nbcg+mnbch=madg+mnadh$,

$\therefore$ $(nbc-mad)g=mn(ad-bc)h$............(4).

Also $pdxg+npdxh=nceg+npceh$,

$\therefore$ $(pdx-nce)g=np(ce-dx)h$, ...(5).

$$\therefore\ \frac{pdx-nce}{nbc-mad}=\frac{p(ce-dx)}{m(ad-bc)},$$

$$\therefore\ pdx(mbc-mad-nbc+mad)=ce(mpad-mnad+mnbc-npbc),$$

$$pdx\{(m-n)bc\}=ce\{(m-p)nbc-(n-p)mad\},$$

$$\therefore\ x=\frac{m-p}{m-n}\cdot\frac{nce}{pd}-\frac{n-p}{m-n}\cdot\frac{mae}{pb}=\frac{e}{p}\left\{\frac{m-p}{m-n}\cdot\frac{nc}{d}-\frac{n-p}{m-n}\cdot\frac{ma}{b}\right\}.$$

Prob. 68. The distance between two places is a, and on the first day $\frac{1}{m}$th of the journey from one to the other is performed; on the 2nd day $\frac{1}{n}$th of the remainder; then $\frac{1}{m}$th and $\frac{1}{n}$th of the remainders alternately on succeeding days. Find the distance gone over in $2p$ days.

$$\frac{a}{m} = \text{journey of 1st day,}$$

$$\left(a - \frac{a}{m}\right)\frac{1}{n}, \text{ or } a\left(1 - \frac{1}{m}\right)\frac{1}{n} = \text{2nd day's,}$$

$$\therefore \text{ journey of 1st pair of days} = \frac{a}{m} + a\left(1 - \frac{1}{m}\right)\frac{1}{n},$$

$$\text{and remainder after} \ldots\ldots\ldots\ldots\ldots = a\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right) \ldots\ldots (1)$$

Hence also remainder after 2nd pair of days will be the same as (1), writing $a\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right)$ for a,

$$= a\left(1 - \frac{1}{m}\right)^2\left(1 - \frac{1}{n}\right)^2; \text{ and so on.}$$

$$\therefore \text{ Remainder after } p \text{ pairs of days} = a\left(1 - \frac{1}{m}\right)^p\left(1 - \frac{1}{n}\right)^p,$$

$$\therefore \text{ journey performed in } 2p \text{ days} = a - a\left(1 - \frac{1}{m}\right)^p\left(1 - \frac{1}{n}\right)^p,$$

$$= a\left\{1 - \left(1 - \frac{1}{m}\right)^p\left(1 - \frac{1}{n}\right)^p\right\}.$$

Prob. 70. To complete a certain work A requires m times as long a time as B and C together; B requires n times as long as A and C together; and C requires p times as long as A and B together. Compare the times in which each would do it; and prove that

$$\frac{1}{m+1} + \frac{1}{n+1} + \frac{1}{p+1} = 1.$$

Let $x = A$'s time of doing the work,

$y = B$'s......................

$z = C$'s......................

$w =$ the work.

Then $\frac{w}{x} = A$'s daily work, $\frac{w}{y} = B$'s, $\frac{w}{z} = C$'s;

and since B and C working together can do m times as much per day as A,

$$\frac{mw}{x} = \frac{w}{y} + \frac{w}{z},$$

$$\left.\begin{aligned} \text{or } \frac{m}{x} &= \frac{1}{y} + \frac{1}{z}. \;\ldots\ldots\ldots(1) \\ \text{Similarly, } \frac{n}{y} &= \frac{1}{x} + \frac{1}{z}, \;\ldots\ldots\ldots(2) \\ \text{and } \frac{p}{z} &= \frac{1}{x} + \frac{1}{y}. \;\ldots\ldots\ldots(3) \end{aligned}\right\}$$

From (1) and (2), $\quad \frac{m}{x} - \frac{n}{y} = \frac{1}{y} - \frac{1}{x},$

$$\therefore \;(m+1)\frac{1}{x} = (n+1)\frac{1}{y}, \text{ or } \frac{x}{y} = \frac{m+1}{n+1} = \frac{A\text{'s time}}{B\text{'s } \ldots}.$$

$$\text{Similarly, } \frac{x}{z} = \frac{m+1}{p+1} = \frac{A\text{'s time}}{C\text{'s } \ldots},$$

$$\text{and } \frac{y}{z} = \frac{n+1}{p+1} = \frac{B\text{'s time}}{C\text{'s } \ldots}.$$

Again, from (1), $\quad m = \frac{x}{y} + \frac{x}{z} = \frac{m+1}{n+1} + \frac{m+1}{p+1},$

$$\therefore \;\frac{1}{n+1} + \frac{1}{p+1} = \frac{m}{m+1},$$

$$= 1 - \frac{1}{m+1},$$

$$\therefore \;\frac{1}{m+1} + \frac{1}{n+1} + \frac{1}{p+1} = 1.$$

PROB. 71. S_1, S_2, S_3, ... S_{n+1} are $(n+1)$ stones placed in a straight line a yard from each other, and X is another assumed station in the same line produced: two persons set out from S_1, the one to carry the stones separately to S_1, the other to X; find the distance from S_{n+1} to X, that the latter may travel exactly twice as far as the former.

Let A and B be the two persons, $S_1 \quad S_2 \quad S_3 \quad S_{n+1} \quad X$

$$\begin{array}{rl} \text{Then the distance } A \text{ travels for 1st stone} & = 0 \text{ yards}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \text{ 2nd } \ldots & = 2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \text{ 3rd } \ldots & = 2\times 2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \text{ 4th } \ldots & = 2\times 3, \\ \ldots\ldots\ldots\ldots\ldots \ (n+1)\text{th } \ldots & = 2\times n, \end{array}$$

$\therefore$ A travels in all $2(1+2+3+\ldots+n)$ yards,

$$\text{or } 2\left(n.\frac{n+1}{2}\right) \text{ yards, (see Art. 282)},$$

$$\text{or } n(n+1) \quad \ldots$$

Let the distance from S_{n+1} to $X=x$,

then B travels $(n+x)+2\{(n+x-1)+(n+x-2)+\text{\&c. to } n \text{ terms}\}$,

$$\text{or } (n+x)(1+2n)-2(1+2+3+\text{\&c. to } n \text{ terms}),$$

$$\text{or } (n+x)(1+2n)-n(n+1),$$

$\therefore$ by the question, $(n+x)(1+2n)-n(n+1)=2n(n+1)$,

$$(n+x)(1+2n)=3n(n+1),$$

$$n+x=\frac{3n(n+1)}{2n+1},$$

$$\therefore x=\frac{3n(n+1)}{2n+1}-n=\frac{n(n+2)}{2n+1}.$$

Prob. 73. Two clocks are striking the hour together, and are heard to strike 19 times. There is a difference of two seconds in their time, and one strikes every three, the other every four, seconds. What is the hour they strike? it being observed that, when the clocks strike in the same second, the sounds cannot be distinguished, so as to determine whether one or both strike in that second, and that this is the case with the last stroke of the faster clock.

Let x be the hour required; then the faster clock, that is, the clock which struck first, and strikes every 3 seconds, was striking the 3rd time, when the other, which strikes every 4 seconds, and is 2 seconds behind, was striking the second time; and this is the 1st time of their striking together.

Afterwards, *every* 4th stroke of the faster clock will coincide with a stroke of the other, since 4×3 seconds $= 3\times 4$ seconds;

and $x-3=$ number of strokes remaining to the faster clock after they had once struck together.

$\therefore\ \frac{x-3}{4}+1=$ number of times in all they strike together,

and $2x=$ number of strokes actually made by both clocks,

$$\therefore\ 2x-\left(\frac{x-3}{4}+1\right)=19,$$

$$\therefore\ \frac{7x}{4}=19+\frac{1}{4}=\frac{77}{4};$$

$$\therefore\ x=11.$$

Ex. 75. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's, but four of B's throw out as much water as five of A's. Now B works for the time in which A alone would have emptied the hold. A then pumps out the remainder, and the hold is cleared in 13hrs. 20min. Had they worked together, the hold would have been emptied in 3hrs. 45min., and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Let $x=$ number of gallons in the hold at first,

$y=$ horary influx at the leak,

$t=$ time, in hours, in which A alone would empty the hold,

$z=$ quantity pumped by A at each stroke,

then $5z=4$ times B's quantity per stroke,

$\therefore\ \frac{5z}{4}=$ quantity pumped by B at each stroke.

But A takes 3 strokes to 2 of B's, therefore while A pumps $3z$ gallons, B pumps $\frac{5z}{4}\times 2$, or $\frac{5z}{2}$, which is $\frac{5}{6}$ of A's work in the same time; and the work is uniform,

$$\therefore\ B\text{'s work } \textit{per hour}=\frac{5}{6}\text{ of } A\text{'s}.$$

Now A would pump out $x+ty$ in time t;

$\therefore$ B would pump out $\frac{5}{6}(x+ty)$ in time $t = B$'s actual work.

And the time A works is $13\frac{1}{3}-t$, and $\frac{x+ty}{t} = A$'s work per hour,

$$\therefore \frac{x+ty}{t}.(13\tfrac{1}{3}-t) = A\text{'s actual work},$$

$$\therefore \frac{5}{6}(x+ty)+(x+ty).\frac{13\frac{1}{3}-t}{t} = x+\frac{40}{3}y,$$

$$\text{or } \left(\frac{40}{3t}-\frac{1}{6}\right)(w+t) = w+\frac{40}{3}, \text{ if } \frac{x}{y}=w,$$

$$\left(\frac{40}{3t}-\frac{7}{6}\right)w = \frac{t}{6},$$

$$\therefore w = \frac{t^2}{80-7t}, \text{ or } \frac{t}{w} = \frac{80-7t}{t} \ldots\ldots\ldots\ldots\ldots(1).$$

Again, if A and B work together, $x+3\frac{3}{4}.y$ clears the hold and A's part is $\frac{6}{11}$ of this,

$$\therefore \frac{6}{11}\left(x+\frac{15y}{4}\right) = \frac{15}{4}.\frac{x+ty}{t},$$

$$\frac{6t}{11}\left(w+\frac{15}{4}\right) = \frac{15}{4}(w+t),$$

$$\frac{15t}{4}.\frac{5}{11} = \left(\frac{6t}{11}-\frac{15}{4}\right)w,$$

$$25t = (8t-55)w, \therefore \frac{t}{w} = \frac{8t-55}{25} \ldots\ldots\ldots\ldots(2).$$

$$\therefore \text{ equating (1) and (2), } \frac{80-7t}{t} = \frac{8t-55}{25},$$

$$2000-175t = 8t^2-55t,$$

$$8t^2+120t = 2000,$$

$$4t^2+60t = 1000,$$

$$4t^2+60t+(15)^2 = 1225,$$

$$2t = -15 \pm 35 = 20, \text{ or } -45;$$

$$\therefore t = 10,$$

$$\text{and } w = \frac{t^2}{80-7t} = \frac{100}{80-70} = 10 = \frac{x}{y}.$$

Again, *A*'s work on the 2nd supposition is 100 gallons more than on the first,

$$\therefore \frac{6}{11}\left(x+\frac{15y}{4}\right)=100+\frac{x+ty}{t}\left(\frac{40}{3}-t\right),$$

$$\frac{6}{11}\left(10y+\frac{15y}{4}\right)=100+2y\left(\frac{40}{3}-10\right),$$

$$\left(\frac{6}{11}\times\frac{55}{4}-\frac{20}{3}\right)y=100,$$

$$\left.\begin{array}{r}\frac{5}{6}y=100, \quad \therefore\ y=120; \\ \text{and } x=10y=1200.\end{array}\right\}$$

INEQUALITIES.

Ex. 4. Shew that $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, if $a+b$ be positive, unless $a=b$.

$$\frac{a}{b^2}+\frac{b}{a^2}-\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{a^2b^2}\{a^3+b^3-ab^2-a^2b\},$$

$$=\frac{1}{a^2b^2}\{(a+b)(a^2-ab+b^2)-(a+b)ab\},$$

$$=\frac{a+b}{a^2b^2}\cdot(a-b)^2.$$

Now $(a-b)^2$ is always positive, and so also is a^2b^2, $\therefore$ the whole quantity is positive, if $a+b$ be positive, that is,

$$\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}, \text{ if } a+b \text{ be positive, unless } a=b.$$

Ex. 6. Shew that $\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}>a$, if $a>b$.

$$\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}>a,$$

$$\text{if } \sqrt{a^2-(a-b)^2}>a-\sqrt{a^2-b^2},$$

$$\text{squaring, if } a^2-a^2+2ab-b^2>a^2-2a\sqrt{a^2-b^2}+a^2-b^2,$$

$$2ab>2a^2-2a\sqrt{a^2-b^2},$$

$$b>a-\sqrt{a^2-b^2},$$

$$\sqrt{a^2-b^2} > a-b,$$
$$a^2-b^2 > a^2-2ab+b^2,$$
$$2ab > 2b^2,$$
$$a > b.$$

Ex. 7. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy or $ac+bd$?

$$x^2y^2-(ac+bd)^2=(a^2+b^2)(c^2+d^2)-(ac+bd)^2,$$
$$=a^2d^2+b^2c^2-2abcd,$$
$$=(ad-bc)^2,$$

$$\therefore\ xy-(ac+bd)=\frac{(ad-bc)^2}{xy+ac+bd}=\text{a positive quantity, unless } ad=bc,$$

$$\therefore\ xy>ac+bd, \text{ unless } ad=bc.$$

Ex. 8. Which is greater n^3+1, or n^2+n?

$$n^3+1-(n^2+n)=n^2(n-1)-(n-1),$$
$$=(n^2-1)(n-1),$$
$$=(n-1)^2(n+1)=\text{a pos. quantity, unless } n=1.$$
$$\therefore\ n^3+1>n^2+n, \text{ unless } n=1.$$

Ex. 9. Which is greater $3(1+a^2+a^4)$ or $(1+a+a^2)^2$?

$$3(1+a^2+a^4)-(1+a+a^2)^2=3+3a^2+3a^4-(1+2a+3a^2+2a^3+a^4),$$
$$=2-2a-2a^3+2a^4,$$
$$=2(1-a)-2a^3(1-a),$$
$$=2(1-a)(1-a^3).$$

Now $1-a$, and $1-a^3$, have the same sign whether $a>$ or <1; and $\therefore$ their product is positive. Hence

$$3(1+a^2+a^4)>(1+a+a^2)^2, \text{ unless } a=1.$$

Ex. 11. Shew that $b\left(\frac{a^2}{b^2}+\frac{b}{a}+2\right)>$ or $<a\left(\frac{b^2}{a^2}+\frac{a}{b}+2\right)$, according as $a>$ or $<b$.

$$b\left(\frac{a^2}{b^2}+\frac{b}{a}+2\right)-a\left(\frac{b^2}{a^2}+\frac{a}{b}+2\right)=\frac{1}{a^2b^2}(a^5+ab^4+2a^3b^3-b^5-a^4b-2a^3b^3),$$

$$=\frac{1}{a^2b^2}\{(a-b)a^4+(a-b)b^4-(a-b)2a^2b^2\},$$

$$=\frac{a-b}{a^2b^2}(a^2-b^2)^2.$$

Now $(a^2-b^2)^2$ is always positive; and so also is a^2b^2, $\therefore$ the whole quantity is positive or negative, according as $a-b$ is positive or negative, that is, according as $a >$ or $< b$.

Ex. 12. Shew that $\frac{n^2-n+1}{n^2+n+1}$ lies between 3 and $\frac{1}{3}$ for all real values of n.

First, $3(n^2-n+1)-(n^2+n+1)=2n^2-4n+2$,

$$=2(n-1)^2,$$

$$=\text{a positive quantity},$$

$\therefore\ 3(n^2-n+1)>n^2+n+1$,

$\therefore\ \frac{n^2-n+1}{n^2+n+1}>\frac{1}{3}$, since n^2+n+1 is positive, Art. 219.

2ndly. $3(n^2+n+1)-(n^2-n+1)=2n^2+4n+2$,

$$=2(n+1)^2,$$

$$=\text{a positive quantity},$$

$$\therefore\ 3(n^2+n+1)>n^2-n+1,$$

$$3>\frac{n^2-n+1}{n^2+n+1},$$

$$\therefore\ \frac{n^2-n+1}{n^2+n+1}<3, \text{ and } >\frac{1}{3};\ \text{or } =\frac{1}{3}, \text{ if } n=1.$$

Ex. 13. Shew that $abc>(a+b-c)(a+c-b)(b+c-a)$, unless $a=b=c$.

$a^2>a^2-(b-c)^2$, since $(b-c)^2$ is necessarily positive,

$\quad >(a+b-c)(a+c-b)$,

$b^2>b^2-(a-c)^2$,

$\quad >(a+b-c)(b+c-a)$,

$c^2>c^2-(a-b)^2$,

$\quad >(a+c-b)(b+c-a)$,

$\therefore\ a^2b^2c^2 > (a+b-c)^2(a+c-b)^2(b+c-a)^2,$

and both sides of the inequality are necessarily positive, (Art. 220.)

$$\therefore\ abc > (a+b-c)(a+c-b)(b+c-a).$$

Second Method.

Let a be the least of the three quantities a, b, c,

and let $b = a+x$,

$c = a+y$,

then $abc = a(a+x)(a+y) = a^3 + a^2(x+y) + axy$ ....(i)

$$\begin{aligned}(a+b-c)(a+c-b)(b+c-a) &= (a+x-y)(a-x+y)(a+x+y),\\ &= (a^2 - \overline{x-y}|^2)(a+x+y),\\ &= a^3 + a^2(x+y) - (a+x+y)(x-y)^2 \text{......(ii)}\end{aligned}$$

$\therefore$ (i)−(ii) $= axy + (a+x+y)(x-y)^2$, which cannot be negative; and cannot be equal to 0, unless $x = y = 0$.

$$\therefore\ abc > (a+b-c)(a+c-b)(b+c-a), \text{ unless } a = b = c.$$

Ex. 14. Shew that $abc > (2a-b)(2b-c)(2c-a)$, unless $a=b=c$, each of the factors being positive.

The inequality holds only when each of the binomial factors is a *positive* quantity. On this supposition, a, b, c, being unequal and positive,

$$\text{let } \left.\begin{aligned} b &= a+x \\ c &= a+y \end{aligned}\right\} \quad \text{then } \left.\begin{aligned} 2a-b &= a-x, \\ 2b-c &= a+2x-y, \\ 2c-a &= a+2y, \end{aligned}\right\}$$

$$\begin{aligned}\therefore\ abc-(2a-b)(2b-c)(2c-a) &= a(a+x)(a+y) - (a-x)(a+2x-y)(a+2y),\\ &= a^3 + a^2(x+y) + axy - a^3 - a^2(x+y)\\ &\quad - 3axy + 2ax^2 + 2ay^2 + 4x^2y - 2xy^2,\\ &= 2a(x^2 - xy + y^2) - 2xy(y-2x),\end{aligned}$$

$\therefore$ the inequality holds, if $a(x^2-xy+y^2) > x(y^2-2xy)$,

but $2a-b = a-x =$ a positive quantity, $\therefore\ a > x$; and plainly

$$x^2 - xy + y^2 > y^2 - 2xy, \quad \therefore\ a(x^2-xy+y^2) > x(y^2-2xy).$$

Hence the proposed inequality holds true.

NOTE. The student will perceive that one of the three factors $2a-b$, $2b-c$, $2c-a$, is *necessarily* positive, and that if only one of the other two be negative, the inequality holds by the rule of signs without further development. But if both are negative, he will be able to shew without much difficulty, that abc may be equal to, greater than, or less than, $(2a-b)(2b-c)(2c-a)$; for example, let $a=1$, $b=4$, $c=10$.

Ex. 15. Shew that $ab(a+b)+ac(a+c)+bc(b+c)$ is between $6abc$, and $2(a^3+b^3+c^3)$, a, b, c, being positive quantities.

The same supposition being made as before,

$$ab(a+b)=a(a+x)(2a+x)=2a^3+3a^2x+ax^2,$$

$$ac(a+c)=a(a+y)(2a+y)=2a^3+3a^2y+ay^2,$$

$$bc(b+c)=(a+x)(a+y)(2a+x+y)=2a^3+3a^2(x+y)+a(x+y)^2+(x+y)xy,$$

$$\therefore\ \text{Sum}=6a^3+6a^2(x+y)+2a(x^2+y^2)+2axy+(x+y)xy\ldots\ldots\ldots\ldots(\text{i})$$

$$6abc=6a^3+6a^2(x+y)+6axy\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii})$$

$$\therefore\ (\text{i})-(\text{ii})=2a(x^2-2xy+y^2)+(x+y)xy,$$

$$=2a(x-y)^2+(x+y)xy, \text{ which is always positive.}$$

Again, $2(a^3+b^3+c^3)=2a^3+2(a^3+3a^2x+3ax^2+x^3)+2(a^3+3a^2y+3ay^2+y^3)$,

$$=6a^3+6a^2(x+y)+6a(x^2+y^2)+2(x^3+y^3)\ldots\ldots\ldots(\text{iii})$$

$$\therefore\ (\text{iii})-(\text{i})=4a(x^2+y^2)-2axy+2(x^3+y^3)-(x+y)xy,$$

$$=4a(x-y)^2+6axy+(x+y)\{2(x^2-xy+y^2)-xy\},$$

$$=4a(x-y)^2+6axy+(x+y)\{2(x-y)^2+xy\}=\text{a positive quantity.}$$

$\therefore$ (i)>(ii) and <(iii), i.e. (i) lies between (ii) and (iii).

Ex. 16. Shew that $(a+b+c)^3>27abc$, and $<9(a^3+b^3+c^3)$, unless $a=b=c$.

The same supposition being made as before,

$$(a+b+c)^3=(3a+x+y)^3=27a^3+27a^2(x+y)+9a(x+y)^2+(x+y)^3\ldots\ (\text{i})$$

$$27abc=27\{a(a+x)(a+y)\}=27a^3+27a^2(x+y)+27axy\ldots\ldots\ldots(\text{ii})$$

$$\therefore (\text{i})-(\text{ii})=9a(x+y)^2-27axy+(x+y)^3,$$

$$=9a(x-y)^2+9axy+(x+y)^3=\text{a positive quantity.}$$

$$\therefore\ (a+b+c)^3>27abc.$$

Again, $9(a^3+b^3+c^3)=9(a^3+a^3+3a^2x+3ax^2+x^3+a^3+3a^2y+3ay^2+y^3)$,

$$=27a^3+27a^2(x+y)+27a(x^2+y^2)+9(x^3+y^3)\ldots(\text{iii})$$

$\therefore$ (iii)$-$(i)$=27a(x^3+y^3)-9a(x+y)^2+9(x^3+y^3)-(x+y)^3,$

$=9a(2x^2+2y^2-2xy)+(x+y)(8x^2+8y^2-7xy),$

$=18a(x-y)^2+18xy+(x+y)\{8(x-y)^2+9xy\}$ = a positive quantity.

$$\therefore\ (a+b+c)^3<9(a^3+b^3+c^3).$$

Ex. 17. If $a<x$, shew that $(x+a)^3-x^3<7ax^2$.

$$(x+a)^3-x^3=3a^2x+3ax^2+a^3,$$

$$=3ax^2+3ax^2.\frac{a}{x}+ax^2.\frac{a^2}{x^2}.$$

Now $a<x$, $\therefore\ \frac{a}{x}<1$, and *a fortiori* $\frac{a^2}{x^2}<1$; $\therefore\ 3ax^2.\frac{a}{x}<3ax^2$, and $ax^2.\frac{a^2}{x^2}<ax^2$.

$$\therefore\ (x+a)^3-x^3<3ax^2+3ax^2+ax^2<7ax^2.$$

Ex. 18. Shew that $\sqrt[3]{n}>\sqrt[4]{n+1}$, for all values of n not less than 3.

$n^4-(n+1)^3=n^4-n^3-3n^2-3n-1,$

$=n^3(n-3)+2n^2(n-3)+3n(n-3)+6(n-3)+17,$

which is always positive, if n is not less than 3,

$\therefore\ n^4>(n+1)^3$, for all values of n not less than 3,

$n^{\frac{4}{12}}>(n+1)^{\frac{3}{12}}$, since both sides are positive, Art. 220,

$n^{\frac{1}{3}}>(n+1)^{\frac{1}{4}},$

or $\sqrt[3]{n}>\sqrt[4]{n+1}$.

Ex. 19. Shew that $(a^2+b^2+1)(c^2+d^2+1)>(ac+bd+1)^2$, unless $a=c$, and $b=d$.

$(a^2+b^2+1)(c^2+d^2+1)-(ac+bd+1)^2=a^2c^2+a^2d^2+a^2+b^2c^2+b^2d^2+b^2+c^2+d^2+1$

$-(a^2c^2+b^2d^2+2ac+2bd+2abcd+1),$

$=a^2d^2+b^2c^2+a^2+b^2+c^2+d^2-2ac-2bd-2abcd,$

$=(ad-bc)^2+(a-c)^2+(b-d)^2$, which is always positive unless $a=c$, and $b=d$.

$\therefore\ (a^2+b^2+1)(c^2+d^2+1)>(ac+bd+1)^2$, unless $a=c$, and $b=d$.

Ex. 20. Shew that $\frac{1}{4}(a+b+c+d) > \sqrt[4]{abcd}$, unless $a=b=c=d$.

This may be proved by the method employed before in Exs. 14, 15, and 16: or it may be done as follows:

$$\left(\frac{A+B}{2}\right)^2 = AB + \left(\frac{A-B}{2}\right)^2,$$

$$> AB.$$

Write $\frac{a+b}{2}$ for A, $\frac{c+d}{2}$ for B, then

$$\left\{\frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right)\right\}^2 > \frac{a+b}{2}\cdot\frac{c+d}{2},$$

$$\therefore \left\{\frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right)\right\}^4 > \left(\frac{a+b}{2}\right)^2\cdot\left(\frac{c+d}{2}\right)^2.$$

$$\text{But } \left(\frac{a+b}{2}\right)^2 = ab + \left(\frac{a-b}{2}\right)^2,$$

$$> ab.$$

$$\text{Similarly, } \left(\frac{c+d}{2}\right)^2 > cd.$$

$$\therefore \left\{\frac{1}{4}(a+b+c+d)\right\}^4 > abcd,$$

$$\therefore \frac{1}{4}(a+b+c+d) > \sqrt[4]{abcd}.$$

Cor. Let $\frac{a+b+c}{3} = d$, then $a+b+c+d = 4d$,

$$\therefore d^4 > abcd,$$

$$d^3 > abc,$$

$$\frac{1}{3}(a+b+c) > \sqrt[3]{abc},$$

or $(a+b+c)^3 > 27abc$, which is Ex. 16.

Ex. 21. If $a_1, a_2, a_3, \ldots a_n$ be positive quantities, shew that

$$\frac{n-1}{2}(a_1+a_2+a_3+\ldots+a_n) > \sqrt{a_1a_2}+\sqrt{a_1a_3}+\sqrt{a_2a_3}+\text{\&c.}$$

$$(\sqrt{a_1}-\sqrt{a_2})^2 = a_1+a_2-2\sqrt{a_1a_2} = \text{a positive quantity},$$

$$\therefore\ a_1+a_2>2\sqrt{a_1a_2}.$$

$$\text{Similarly,}\quad a_1+a_3>2\sqrt{a_1a_3},$$

$$a_2+a_3>2\sqrt{a_2a_3},$$

$$\ldots\ldots>\ldots\ldots\ldots$$

$$a_{n-2}+a_n>2\sqrt{a_{n-2}a_n},$$

$$a_{n-1}+a_n>2\sqrt{a_{n-1}a_n},$$

$$\therefore\ (n-1)(a_1+a_2+a_3+\ldots+a_n)>2(\sqrt{a_1a_2}+\sqrt{a_1a_3}+\sqrt{a_2a_3}+\&c.),$$

$$\therefore\ \frac{n-1}{2}(a_1+a_2+a_3+\ldots+a_n)>(\sqrt{a_1a_2}+\sqrt{a_1a_3}+\sqrt{a_2a_3}+\&c.).$$

Ex. 22. Shew that $\dfrac{1}{a^n+a^{-n}-1}<1$, for all *real* values of a.

Multiplying Num^r. and Denom^r. by a^n, which will not alter its value, the fraction becomes

$$\frac{a^n}{a^{2n}-a^n+1},\ \text{or}\ \frac{a^n}{(a^n-1)^2+a^n}.$$

Now, the denominator is plainly greater than the numerator, if a be *real;* therefore the fraction is less than 1, or

$$\frac{1}{a^n+a^{-n}-1}<1.$$

Observe, unless a be *real,* the square of a quantity involving a will not necessarily be positive, since $(\sqrt{-1})^2$ is -1; but if a be *real,* then any power or root of a is *real,* and the square of any quantity involving it with or without other *real* quantities must be positive.

N.B. In *all* the preceding Examples of Inequalities the quantities concerned are supposed *real,* that is, not to involve $\sqrt{-1}$.

RATIOS, PROPORTION, AND VARIATION.

Ex. 7. Prove that, if $a : b$ is a greater ratio than $c : d$, $a+c : b+d$ is a less ratio than $a : b$, but a greater than $c : d$.

$$a:b>c:d,\quad \therefore\ \frac{a}{b}>\frac{c}{d},\ \text{and}\ \frac{a}{c}>\frac{b}{d},$$

$$\therefore\ \frac{c}{a}<\frac{d}{b},\quad 1+\frac{c}{a}<1+\frac{d}{b},\quad \text{or}\ \frac{a+c}{a}<\frac{b+d}{b},$$

$$\therefore \frac{a+c}{b+d} < \frac{a}{b}, \quad \text{or } a+c : b+d < a : b.$$

Again, $\frac{a}{c} > \frac{b}{d}$, $\frac{a+c}{c} > \frac{b+d}{d}$, $\frac{a+c}{b+d} > \frac{c}{d}$, $\therefore a+c : b+d > c : d$.

Ex. 19. If $a : b :: c : d$, shew that

$$\frac{1}{ma} + \frac{1}{nb} + \frac{1}{pc} + \frac{1}{qd} = \frac{1}{bc}\left\{\frac{a}{q} + \frac{b}{p} + \frac{c}{n} + \frac{d}{m}\right\}.$$

Since $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$, and $ad = bc$,

$$\therefore a = \frac{bc}{d}, \quad ma = bc \cdot \frac{m}{d}, \quad \frac{1}{ma} = \frac{1}{bc} \cdot \frac{d}{m}.$$

Again, $\frac{1}{nb} = \frac{1}{bc} \cdot \frac{c}{n}$. Also $\frac{1}{pc} = \frac{1}{bc} \cdot \frac{b}{p}$. And $\frac{1}{qd} = \frac{1}{ad} \cdot \frac{a}{q} = \frac{1}{bc} \cdot \frac{a}{q}$,

$$\therefore \frac{1}{ma} + \frac{1}{nb} + \frac{1}{pc} + \frac{1}{qd} = \frac{1}{bc}\left\{\frac{a}{q} + \frac{b}{p} + \frac{c}{n} + \frac{d}{m}\right\}.$$

Ex. 20. If $a_1 : b_1 :: a_2 : b_2 :: a_3 : b_3 ::$ &c., shew that

$$(a_1^2 + a_2^2 + a_3^2 + \&c.)(b_1^2 + b_2^2 + b_3^2 + \&c.) = (a_1b_1 + a_2b_2 + a_3b_3 + \&c.)^2;$$

and $\sqrt{(a_1 + a_2 + a_3 + \&c.)(b_1 + b_2 + b_3 + \&c.)} = \sqrt{a_1b_1} + \sqrt{a_2b_2} + \sqrt{a_3b_3} + \&c.$

Here $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \&c.$ $\therefore \frac{a_1^2}{b_1^2} = \frac{a_2^2}{b_2^2} = \frac{a_3^2}{b_3^2} = \&c.$

$$\therefore a_1^2 : b_1^2 :: a_2^2 : b_2^2 :: a_3^2 : b_3^2 :: \&c.$$

$\therefore a_1^2 : b_1^2 :: a_1^2 + a_2^2 + a_3^2 + \&c. : b_1^2 + b_2^2 + b_3^2 + \&c.$ Art. 248.

$$\therefore (a_1^2 + a_2^2 + a_3^2 + \&c.)(b_1^2 + b_2^2 + b_3^2 + \&c.) = \frac{a_1^2}{b_1^2}(b_1^2 + b_2^2 + b_3^2 + \&c.)^2,$$

$$= \left\{\frac{a_1}{b_1}(b_1^2 + b_2^2 + b_3^2 + \&c.)\right\}^2,$$

$$= (a_1b_1 + \frac{a_2}{b_2}b_2^2 + \frac{a_3}{b_3}b_3^2 + \&c.)^2,$$

$$= (a_1b_1 + a_2b_2 + a_3b_3 + \&c.)^2.$$

The second thing to be proved follows directly from the first, writing $\sqrt{a_1}$, $\sqrt{a_2}$, &c., $\sqrt{b_1}$, $\sqrt{b_2}$, &c. for a_1, a_2, &c., b_1, b_2, &c.

Ex. 21. If the difference between a and b be small when compared with either of them, shew that the ratio $\sqrt[m]{a} - \sqrt[m]{b} : \sqrt[n]{a} - \sqrt[n]{b}$ is nearly equal to $n\sqrt[m]{a} : m\sqrt[n]{a}$.

$$\sqrt[n]{a}-\sqrt[n]{b} : \sqrt[m]{a}-\sqrt[m]{b}=\frac{\sqrt[n]{a}}{\sqrt[m]{a}}\cdot\frac{1-\left(\frac{b}{a}\right)^{\frac{1}{n}}}{1-\left(\frac{b}{a}\right)^{\frac{1}{m}}}=\frac{\sqrt[n]{a}}{\sqrt[m]{a}}\cdot\frac{1-x^n}{1-x^m},\ \text{ if } \left(\frac{b}{a}\right)^{\frac{1}{mn}}=x,$$

$$\left.\begin{array}{l}\text{dividing numerator and}\\ \text{denominator by } 1-x,\end{array}\right\}=\frac{\sqrt[n]{a}}{\sqrt[m]{a}}\cdot\frac{1+x+x^2+\ldots\ldots+x^{n-1}}{1+x+x^2+\ldots\ldots+x^{m-1}},$$

$$\left(\text{if } \frac{b}{a}=1, \text{ i.e. } a=b, \text{ nearly,}\right)=\frac{\sqrt[n]{a}}{\sqrt[m]{a}}\cdot\frac{1+1+1+\&\text{c. to } n \text{ terms}}{1+1+1+\&\text{c. to } m \text{ terms}}=\frac{n\sqrt[n]{a}}{m\sqrt[m]{a}}.$$

$$\therefore \sqrt[n]{a}-\sqrt[n]{b} : \sqrt[m]{a}-\sqrt[m]{b}=n\sqrt[n]{a} : m\sqrt[m]{a}.$$

Ex. 22. If $\dfrac{\sqrt[n]{x}-m\sqrt[n]{y}}{\sqrt[n]{x}+m\sqrt[n]{y}}=\dfrac{\sqrt[2n]{x}-m\sqrt[2n]{x-y}}{\sqrt[2n]{x}+m\sqrt[2n]{x-y}}$, shew that $x : y :: 1\pm\sqrt{5} : 2$.

By Art. 195, $\dfrac{\sqrt[n]{x}}{m\sqrt[n]{y}}=\dfrac{\sqrt[2n]{x}}{m\sqrt[2n]{x-y}}$,

$$\therefore \frac{x^2}{y^2}=\frac{x}{x-y}, \text{ or } \frac{x}{y^2}=\frac{1}{x-y},$$

$$\therefore \frac{x^2}{y^2}-\frac{x}{y}=1,$$

$$\frac{x^2}{y^2}-\frac{x}{y}+\frac{1}{4}=\frac{5}{4},$$

$$\frac{x}{y}-\frac{1}{2}=\pm\frac{\sqrt{5}}{2},$$

$$\frac{x}{y}=\frac{1\pm\sqrt{5}}{2},$$

$$\text{or } x : y :: 1\pm\sqrt{5} : 2.$$

Ex. 26. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver.

Let D be the diameter of a sovereign, d of a shilling.

T ...... thickness .... t ....

Then by the question, $md=nD$, and $pt=qT$.

But quantity of gold in a sovereign : do. of silver in a shilling

$$:: D^2T : d^2t, \because \text{circles} \propto (\text{diam.})^2.$$

$$:: \frac{D^2}{d^2} : \frac{t}{T},$$

$$:: \frac{m^2}{n^2} : \frac{q}{p}, \quad \because \frac{D}{d} = \frac{m}{n}, \text{ and } \frac{t}{T} = \frac{q}{p}.$$

Also value of a unit of gold × number of units in a sovereign : value of unit of silver × number of units in a shilling :: 20 : 1.

∴ value of a unit of gold : do. of silver

:: 20 × number of units in a shilling

: number of units in a sovereign,

$$:: \frac{20q}{p} : \frac{m^2}{n^2},$$

$$:: 20n^2q : m^2p.$$

Ex. 27. A person in a railway carriage observes that another train running on a parallel line in the opposite direction occupies two seconds in passing him; but, if the two trains had been proceeding in the same direction, it would have taken 30 seconds to pass him. Compare the speeds of the two trains.

Let x be the speed of the quicker train, and y of the slower, measured by the space passed over in 1 second. Then, in 2 seconds, on the 1st supposition, one train passes over $2x$, and the other $2y$, and

$$2x + 2y = \text{length of the quicker train.}$$

Again, on the 2nd supposition,

$$30x - 30y = \text{length of the quicker train;}$$

$$\therefore 30x - 30y = 2x + 2y,$$

$$28x = 32y;$$

$$\therefore 7x = 8y, \text{ and } x : y :: 8 : 7.$$

Ex. 28. A person, having travelled 56 miles on a railway, and the rest of his journey by a coach, observed that in the train he had performed one-fourth of his whole journey in the time the coach took to go 5 miles, and that at the instant he arrived at home the train would be 35 miles further than he was from the station at which it left him. Compare the rates of the coach and train.

Let the part of the journey by the coach be $4x$ miles; then the whole journey was $56 + 4x$ miles, and one-fourth of it was

$14+x$ miles. The coach takes the same *time* to go 5 miles as the train does to go $14+x$ miles; $\therefore$ rate of coach : rate of train $:: 5 : 14+x$.

Again, while the coach travels $4x$ miles, the train travels $35+4x$ miles; so that

$$\text{rate of coach} : \text{rate of train} :: 4x : 35+4x.$$

Hence $4x : 35+4x :: 5 : 14+x$,

$$\therefore\ 4x(14+x)=5(35+4x),$$

$$4x^2+56x=175+20x,$$

$$4x^2+36x+81=256,$$

$$2x+9=16,$$

$$\therefore\ x=3\tfrac{1}{2}.$$

Hence rate of coach : rate of train $:: 5 : 17\frac{1}{2} :: 1 : 3\frac{1}{2}$.

Ex. 34. Given that $x \propto \dfrac{1}{y^m}$, and $y \propto \dfrac{1}{z^n}$; also when $x=a$, $z=c$; find the equation between x and z.

Let $x=\dfrac{p}{y^m}$, $y=\dfrac{q}{z^n}$, p and q being constants,

then $x=\dfrac{p}{q^m}z^{mn}$, and $a=\dfrac{p}{q^m}c^{mn}$, $\therefore\ \dfrac{p}{q^m}=\dfrac{a}{c^{mn}}$,

$$\therefore\ x=\frac{a}{c^{mn}}.z^{mn},\ \text{ or }\ az^{mn}=c^{mn}x.$$

Ex. 35. If $y=r+s$, whilst $r\propto x$, and $s\propto\sqrt{x}$; and if, when $x=4$, $y=5$, and when $x=9$, $y=10$; find the equation between x and y.

Given $\left.\begin{array}{r} y=r+s, \\ r\propto x, \\ s\propto\sqrt{x}, \end{array}\right\}$ $\therefore\ r=mx$,
$s=n\sqrt{x}$, and $y=mx+n\sqrt{x}$, where m and n are unknown constants.

$$\left.\begin{array}{l} \text{If } \left.\begin{array}{r} x=4, \\ y=5, \end{array}\right\} \left.\begin{array}{l} r=4m, \\ s=2n, \end{array}\right\} \text{ and } \therefore\ 5=4m+2n,\ldots(1) \\ \text{If } \left.\begin{array}{r} x=9, \\ y=10, \end{array}\right\} \left.\begin{array}{l} r=9m, \\ s=3n, \end{array}\right\} \text{ and } \therefore\ 10=9m+3n,\ldots(2) \end{array}\right\} \text{ to find } m \text{ and } n.$$

$$\left.\begin{array}{l}\text{From (1), } 12m+6n=15\\ \quad\ldots\quad (2),\ 18m+6n=20\end{array}\right\},$$

$$\therefore\ 6m=5,$$

$$m=\frac{5}{6}.$$

Also $2n=5-4m=5-\frac{10}{3}=\frac{5}{3}$, $\therefore\ n=\frac{5}{6}$.

$$\therefore\ y=\frac{5}{6}(x+\sqrt{x}).$$

Ex. 40. If $A\propto B$, and $B\propto C$, shew that

$$mA+nB+pC\propto m'\sqrt{AB}+n'\sqrt{BC}+p'\sqrt{AC}.$$

Since $\left.\begin{array}{ll}A\propto B, & A=rB,\\ B\propto C, & B=sC,\end{array}\right\}$ r and s being constants,

$$rB^2=AB,\quad sC^2=BC,\quad rsC^2=AC,$$

$$\therefore\ mA=mrB=m\sqrt{r}.\sqrt{AB},$$

$$nB=nsC=n\sqrt{s}.\sqrt{BC},$$

$$pC=\frac{p}{\sqrt{rs}}\sqrt{AC},$$

$$\therefore\ mA+nB+pC=m\sqrt{r}.\sqrt{AB}+n\sqrt{s}.\sqrt{BC}+\frac{p}{\sqrt{rs}}\sqrt{AC},$$

$$=c\left(\frac{m\sqrt{r}}{c}\sqrt{AB}+\frac{n\sqrt{s}}{c}\sqrt{BC}+\frac{p}{c\sqrt{rs}}\sqrt{AC}\right),$$

$$\propto m'\sqrt{AB}+n'\sqrt{BC}+p'\sqrt{AC},\ c \text{ being constant,}$$

and $m'=\frac{m\sqrt{r}}{c}$, $n'=\frac{n\sqrt{s}}{c}$, $p'=\frac{p}{c\sqrt{rs}}$.

Ex. 41. If $A\propto B$, and $B\propto\sqrt{AC}$, shew that $C\propto\sqrt[3]{A^2B}+\sqrt[3]{AB^2}$, and that $m\sqrt[4]{AB}-n\sqrt[4]{BC}\propto p\sqrt{A}+q\sqrt{B}$.

Since $\left.\begin{array}{ll}A\propto B, & A=rB,\\ B\propto\sqrt{AC}, & B=s\sqrt{AC},\end{array}\right\}$ r and s being constants.

Then $A = rs\sqrt{AC}$, or $A = r^2s^2C$, and $B = \frac{A}{r} = rs^2C$,

$$\therefore\ A^2B = r^5s^6C^3, \text{ and } AB^2 = r^4s^6C^3,$$

$$\therefore\ \sqrt[3]{A^2B} + \sqrt[3]{AB^2} = (r^{\frac{5}{3}}s^2 + r^{\frac{4}{3}}s^2)\,C \propto C,$$

$$\text{or } C \propto \sqrt[3]{A^2B} + \sqrt[3]{AB^2}.$$

$$\text{Again, } AB = \frac{1}{r}A^2, \quad \therefore\ m\sqrt[4]{AB} = m\sqrt[4]{\frac{1}{r}}.\sqrt{A},$$

$$BC = \frac{1}{r}AC = \frac{1}{rs^2}.B^2, \quad \therefore\ -n\sqrt[4]{BC} = -n\sqrt[4]{\frac{1}{rs^2}}.\sqrt{B},$$

$$\therefore\ m\sqrt[4]{AB} - n\sqrt[4]{BC} = m\sqrt[4]{\frac{1}{r}}.\sqrt{A} - n\sqrt[4]{\frac{1}{rs^2}}.\sqrt{B},$$

$$= l\left\{\frac{m}{l}\sqrt[4]{\frac{1}{r}}.\sqrt{A} - \frac{n}{l}\sqrt[4]{\frac{1}{rs^2}}.\sqrt{B}\right\},$$

$$\propto p\sqrt{A} + q\sqrt{B}, \text{ where } p = \frac{m}{l}\sqrt[4]{\frac{1}{r}}, \text{ and } q = \frac{-n}{l}\sqrt[4]{\frac{1}{rs^2}}.$$

Ex. 42. If $mA + nB \propto pA - qB$, and $m'A - n'C \propto p'B + q'C$, shew that $(\alpha_1\sqrt{A} - \beta_1\sqrt{B} + \gamma_1\sqrt{C})^2 \propto \alpha_2A + \beta_2B + \gamma_2C$.

Since $mA + nB \propto pA - qB$, $\therefore\ mA + nB = r(pA - qB)$,

$$\therefore\ (n + rq)B = (rp - m)A, \quad B = \frac{rp - m}{n + rq}.A = sA.$$

Similarly, from 2nd variation, and putting sA for B, $C = tA$.

$$\therefore\ (\alpha_1\sqrt{A} - \beta_1\sqrt{B} + \gamma_1\sqrt{C})^2 = (\alpha_1\sqrt{A} - \beta_1\sqrt{s}.\sqrt{A} + \gamma_1\sqrt{t}.\sqrt{A})^2,$$

$$= (\alpha_1 - \beta_1\sqrt{s} + \gamma_1\sqrt{t})^2A,$$

$$= (h + k + l)\,A,$$

$$= hA + \frac{k}{s}B + \frac{l}{t}C,$$

$$= c\left(\frac{h}{c}A + \frac{k}{cs}B + \frac{l}{ct}C\right),$$

$$\propto \frac{h}{c}A + \frac{k}{cs}B + \frac{l}{ct}C,$$

$$\propto \alpha_2A + \beta_2B + \gamma_2C, \text{ if } \alpha_2 = \frac{h}{c}, \ \beta_2 = \frac{k}{sc}, \ \gamma_2 = \frac{l}{ct}.$$

Ex. 43. A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto$ (radii)^3.

Let R be the outer radius, and W the weight of such a *solid* sphere,

r ...... inner.............. w ....,

$$\therefore W : w :: R^3 : r^3,$$

$$W-w : W :: R^3-r^3 : R^3,$$

$$\frac{7}{8} : 1 :: R^3-r^3 : R^3,$$

$$7 : 8 :: R^3-r^3 : R^3;$$

$$\therefore 1 : 8 :: r^3 : R^3,$$

$$1 : 2 :: r : R.$$

Ex. 45. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 miles an hour. Find the greatest number of waggons which the engine can move.

Let x = the number of waggons attached;

then $24-c\sqrt{x}$ = speed of the train, c being a constant unknown.

$\therefore$ by the question, $20=24-c\sqrt{x}$, when $x=4$,

$$=24-2c, \quad \therefore c=2,$$

and $\therefore 24-2\sqrt{x}$ = the speed with x waggons.

Now if the speed is reduced to 0, $24-2\sqrt{x}=0$,

$$\therefore \sqrt{x}=12, \text{ and } x=144.$$

$\therefore$ Number of waggons required $=143$.

Ex. 46. The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ cube of their weight. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth c £. Required the values of a diamond and of a ruby, each weighing n carats.

The value of a diamond weighing n carats $\propto n^2 = pn^2$, $\Big\}$ p and q
.... ruby .... $\propto n^{\frac{3}{2}} = qn^{\frac{3}{2}}$,

being constants unknown.

$$\therefore \text{ a diamond of } a \text{ carats is worth } pa^2,$$

$$\ldots \text{ ruby } \ldots b \ldots\ldots\ldots\ldots\ldots\ldots \; qb^{\frac{3}{2}},$$

and $\therefore$ by the question, $\left.\begin{array}{r} pa^2 = mqb^{\frac{3}{2}}, \\ \text{also } pa^2 + qb^{\frac{3}{2}} = c, \end{array}\right\}$ to find p and q.

$$mqb^{\frac{3}{2}} + qb^{\frac{3}{2}} = c,$$

$$\therefore \; q = \frac{c}{(m+1)b^{\frac{3}{2}}}.$$

$$\text{Also } mpa^2 + pa^2 = mc,$$

$$\therefore \; p = \frac{mc}{(m+1)a^2}.$$

$$\therefore \text{ Value of the diamond} = pn^2 = \frac{mcn^2}{(m+1)a^2}.$$

$$\text{Value of the ruby} = qn^{\frac{3}{2}} = \frac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}.$$

Ex. 47. If the prices of two trees containing p and q cubic feet of timber be a£ and b£; required the price of a tree containing r cubic feet, supposing the values of the timber and bark to be respectively proportional to the m^{th} and n^{th} powers of the quantity of timber in the tree.

Let x = the number of cubic feet of timber in any tree, then since

the value of the Timber $\propto x^m$, and $\therefore = Tx^m$, T being constant,

and .... Bark $\propto x^n$, and $\therefore = Bx^n$, B .... ,

Whole Value of a Tree $= Tx^m + Bx^n$, where T and B are unknown quantities to be determined.

Now, by the question, $\left.\begin{array}{r} a = Tp^m + Bp^n, \\ b = Tq^m + Bq^n, \end{array}\right\}$

$$\therefore \left.\begin{array}{r} aq^m = Tp^m q^m + Bp^n q^m, \\ bp^m = Tp^m q^m + Bp^m q^n, \end{array}\right\}$$

$$\therefore\ aq^m - bp^m = B(p^nq^m - p^mq^n),$$

$$\therefore\ B = \frac{aq^m - bp^m}{p^nq^m - p^mq^n}.$$

Again, $\left.\begin{aligned} aq^n &= Tp^mq^n + Bp^nq^n, \\ bp^n &= Tp^nq^m + Bp^nq^n, \end{aligned}\right\}$

$$bp^n - aq^n = T(p^nq^m - p^mq^n);$$

$$\therefore\ T = \frac{bp^n - aq^n}{p^nq^m - p^mq^n}.$$

Hence the value required $= Tr^m + Br^n$,

$$= \frac{(bp^n - aq^n)\,r^m + (aq^m - bp^m)\,r^n}{p^nq^m - p^mq^n}.$$

ARITHMETICAL PROGRESSION.

Ex. 15. Find the series in A. P. in which 7 and 5 are the 5th and 7th terms respectively.

Let a, $a+x$, $a+2x$, &c. be the series,

$$\left.\begin{aligned} \text{then } 5^{\text{th}} \text{ term} &= a+4x = 7, \\ \text{and } 7^{\text{th}} \text{ term} &= a+6x = 5, \end{aligned}\right\} \text{ to find } a \text{ and } x,$$

$$2x = -2, \quad \therefore\ x = -1.$$

Then $a + 4x = a - 4 = 7$, $\therefore\ a = 11$.

$\therefore$ 11, 10, 9, 8, 7, 6, 5, &c. is the series required.

Ex. 18. If the Arith^e. Mean between a and b be twice as great as the Geom^e. Mean, shew that $a : b :: 2+\sqrt{3} : 2-\sqrt{3}$.

The Arith^e. Mean $= \dfrac{a+b}{2}$, the Geom^e. Mean $= \sqrt{ab}$,

and by the question, $\dfrac{a+b}{2} = 2\sqrt{ab}$,

$$\therefore\ \frac{a+b}{2\sqrt{ab}} = 2,$$

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{1}, \quad \text{Art. 195.}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{3}}{1},$$

$$\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{3}+1}{\sqrt{3}-1},$$

$$\therefore \frac{a}{b}=\frac{4+2\sqrt{3}}{4-2\sqrt{3}}=\frac{2+\sqrt{3}}{2-\sqrt{3}},$$

$$\text{or } a : b :: 2+\sqrt{3} : 2-\sqrt{3}.$$

Ex. 19. If the Arith^c. Mean between a and b be m times as great as the Harmonic Mean, shew that

$$a : b :: \sqrt{m}+\sqrt{m-1} : \sqrt{m}-\sqrt{m-1}.$$

The Arith^c. Mean $=\frac{a+b}{2}$. The Harmonic Mean $=\frac{2ab}{a+b}$.

And by the question, $\frac{a+b}{2}=m.\frac{2ab}{a+b}$,

$$\therefore \frac{a^2+2ab+b^2}{4ab}=m,$$

$$\frac{a^2+2ab+b^2}{a^2-2ab+b^2}=\frac{m}{m-1}, \quad \text{Art. 195.}$$

$$\frac{a+b}{a-b}=\frac{\sqrt{m}}{\sqrt{m-1}},$$

$$\therefore \frac{a}{b}=\frac{\sqrt{m}+\sqrt{m-1}}{\sqrt{m}-\sqrt{m-1}},$$

$$\text{or } a : b :: \sqrt{m}+\sqrt{m-1} : \sqrt{m}-\sqrt{m-1}.$$

Ex. 20. Shew that if the same number of Arith^c. Means be inserted between every two contiguous terms of an A. P. the whole will be in A. P.

Let a, b, c, d, &c. be a series in A. P., and suppose

n Arith^c. Means $a+x$, $a+2x$, &c...$a+(n-1)x$, inserted between a and b,
.... $b+y$, $b+2y$, &c...$b+(n-1)y$, ... b and c,
.... $c+z$, $c+2z$, &c...$c+(n-1)z$, ... c and d,
&c. &c. &c.

$$\text{Then}\ \left.\begin{array}{r} a+nx=b, \\ b+ny=c, \\ c+nz=d, \\ \&c.\ \&c. \end{array}\right\} \begin{array}{l} \therefore\ a-b+n(x-y)=b-c, \\ \text{but}\ a-b=b-c,\quad \because\ a,\ b,\ c\ \text{are in A. P.} \\ \qquad \therefore\ n(x-y)=0, \\ \qquad\quad \therefore\ x-y=0,\quad \text{or}\ x=y. \end{array}$$

Similarly, $y=z$, and so on.

$$\therefore\ x=y=z=\&c.$$

$\therefore$ the series becomes

$$a,\ a+x,\ \&c.\ b,\ b+x,\ \&c.\ c,\ c+x,\ \&c.\ \text{which is in A. P.}$$

Ex. 24. Divide $\frac{n}{7}(n+4)$ into n parts, such that each part shall exceed the one immediately preceding by a fixed quantity.

The n parts must form an Arith^{c}. Series, whose sum $=\frac{n}{7}(n+4)$.

Let $n=1$, then the series is reduced to one term, which is the sum in that case, $\therefore$ the first term or part $=\frac{5}{7}$.

Let $n=2$, then the sum of 2 terms or parts $=\frac{12}{7}$, $\therefore$ second term $=1$, and the Com. Diff. $=\frac{2}{7}$. Hence the parts required are

$$\frac{5}{7},\ 1,\ 1\tfrac{2}{7},\ 1\tfrac{4}{7},\ \&c.$$

Ex. 25. The n^{th} term of an Arith^{c}. Prog^{n}. is $\frac{1}{6}(3n-1)$, prove that the sum of n terms is $\frac{n}{12}(3n+1)$, and find the series.

Since the n^{th} term $=\frac{3n-1}{6}$, the first term is found by putting 1 for n, and $\therefore$ first term $=\frac{1}{3}$. Hence the sum of n terms $=(a+l)\frac{n}{2}$, (Art. 282)

$$=\left\{\frac{1}{3}+\frac{1}{6}(3n-1)\right\}\frac{n}{2},$$

$$=\frac{n}{12}(3n+1).$$

Again, putting 2 for n, the second term $=\frac{5}{6}$, $\therefore$ Com. diff. $=\frac{3}{6}=\frac{1}{2}$,

$$\text{and } \therefore \text{ the Series is } \frac{1}{3},\ \frac{5}{6},\ \frac{4}{3},\ \frac{11}{6},\ \&c.$$

Ex. 27. There are two series in Arith^e. Prog^n., the sums of which to n terms are as $13-7n : 3n+1$; prove that their first terms are as $3 : 2$, and their second terms as $-4 : 5$.

$$\text{Let } \left.\begin{matrix} A, & A+B, & A+2B, & \&c. \\ a, & a+b, & a+2b, & \&c. \end{matrix}\right\} \text{ be the two series:}$$

Then $\{2A+(n-1)B\}\frac{n}{2} : \{2a+(n-1)b\}\frac{n}{2} :: 13-7n : 3n+1$.

$$\text{Let } n=1, \text{ then } A : a :: 6 : 4 :: 3 : 2 \ldots\ldots\ldots\ldots\ldots (1).$$

Again, let $n=3$, then $2A+2B : 2a+2b :: -8 : 10$,

$$\text{or } A+B \ : \ a+b \ :: -4 : 5 \ldots\ldots\ldots (2).$$

Ex. 32. Determine the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of an Arith^e. Prog^n.

Let A be the first term, and B the Com. diff., of the series; then

$$\left.\begin{aligned} a&=A+(p-1)B, \\ b&=A+(q-1)B, \\ c&=A+(r-1)B, \end{aligned}\right\} \text{ from which equations } A \text{ and } B \text{ are to be eliminated.}$$

$$\left.\begin{aligned} a-b&=(p-q)B, \\ b-c&=(q-r)B, \end{aligned}\right\}$$

$$\therefore \frac{a-b}{b-c}=\frac{p-q}{q-r},$$

$$(q-r)a-(q-r)b=(p-q)b-(p-q)c,$$

$$\therefore (q-r)a+(r-p)b+(p-q)c=0.$$

Ex. 33. In an Arith^e. Prog^n., if the $(p+q)^{\text{th}}$ term $=m$, and the $(p-q)^{\text{th}}$ term $=n$, shew that the p^{th} term $=\frac{1}{2}(m+n)$, and the q^{th} term $=m-(m-n)\frac{p}{2q}$.

Let a be the first term, and b the com. diff.; then

$$\left.\begin{array}{ll} \text{the } (p+q)^{\text{th}} \text{ term} = a+(p+q-1)b = m, & \ldots\ldots\ldots(1) \\ \text{and } (p-q)^{\text{th}} \ldots\ldots = a+(p-q-1)b = n, & \ldots\ldots\ldots(2) \end{array}\right\}$$

$$\therefore\ m+n = 2a+2(p-1)b,$$

$$\frac{1}{2}(m+n) = a+(p-1)b = \text{the } p^{\text{th}} \text{ term.}$$

Again, $m-n=2qb$, $\therefore\ b=\dfrac{m-n}{2q}$, and $pb=(m-n)\dfrac{p}{2q}$,

$$\therefore\ a+(q-1)b = m-pb, \text{ from (1)},$$

$$= m-(m-n)\frac{p}{2q} = \text{the } q^{\text{th}} \text{ term.}$$

Ex. 34. If the m^{th} term of an Arith$^{\text{c}}$. Prog$^{\text{n}}$. $=n$, and the n^{th} term $=m$, of how many terms will the sum be $\frac{1}{2}(m+n)(m+n-1)$; and what will be the last of them?

Let a be the first term, b the com. diff., and x the number of terms, then

$$\left.\begin{array}{l} a+(m-1)b = n, \\ a+(n-1)b = m, \end{array}\right\}$$

$$\therefore\ (m-n)b = n-m = -(m-n), \text{ or } b=-1,$$

$$\text{and } \therefore\ a = m-(n-1)b = m+n-1.$$

Hence, substituting in the general formula $s=\{2a+(x-1)b\}\dfrac{x}{2}$, we have

$$\frac{1}{2}(m+n)(m+n-1) = \{2(m+n-1)-(x-1)\}\frac{x}{2},$$

$$= \left(m+n-\frac{1}{2}\right)x - \frac{x^2}{2},$$

$$x^2 - 2\left(m+n-\frac{1}{2}\right)x = -(m+n)(m+n-1),$$

$$x^2-2\left(m+n-\frac{1}{2}\right)x+\left(m+n-\frac{1}{2}\right)^2 = \left(m+n-\frac{1}{2}\right)^2-(m+n)^2+(m+n),$$

$$= \frac{1}{4},$$

$$\therefore\ x = m+n-\frac{1}{2}\pm\frac{1}{2},$$
$$= m+n, \quad \text{or } m+n-1.$$

Again, the last, or x^{th}, term $= a+(x-1)b$,

$$= m+n-1-(m+n-1), \text{ or } m+n-1-(m+n-2),$$
$$= 0, \text{ or } 1.$$

Ex. 35. The sum of m terms of an Arith$^{\text{c}}$. Prog$^{\text{n}}$. is n, and the sum of n terms is m; shew that the sum of $(m+n)$ terms is $-(m+n)$, and the sum of $(m-n)$ terms is $(m-n)\left(1+\frac{2n}{m}\right)$.

Let a be the first term, and b the com. diff., then, by the question,

$$\left.\begin{aligned} &\{2a+(m-1)b\}\frac{m}{2}=n, \\ \text{and}\quad &\{2a+(n-1)b\}\frac{n}{2}=m, \end{aligned}\right\} \text{ to find } a \text{ and } b.$$

$$\left.\begin{aligned} &2a+(m-1)b=\frac{2n}{m}, \\ \text{and}\quad &2a+(n-1)b=\frac{2m}{n}, \end{aligned}\right\}$$

$$(m-n)b=\frac{2n}{m}-\frac{2m}{n}=\frac{2}{mn}(n^2-m^2),$$

$$\therefore\ b=-\frac{2}{mn}(m+n)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{And } 2a=\frac{2n}{m}+(m-1)\frac{2}{mn}(m+n),$$

$$=\frac{2}{mn}\{n^2+(m-1)(m+n)\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

(1) $\cdot\therefore$ Sum of $(m+n)$ terms

$$=\{2a+(m+n-1)b\}\frac{m+n}{2},$$

$$=\left[\frac{2}{mn}\{n^2+(m-1)(m+n)\}-(m+n-1)\frac{2}{mn}(m+n)\right]\cdot\frac{m+n}{2},$$

$$=\frac{m+n}{mn}\{n^2+(m-1)(m+n)-(m+n-1)(m+n)\},$$

$$=\frac{m+n}{mn}\{n^2-n(m+n)\},$$

$$=-(m+n).$$

(2) Sum of $(m-n)$ terms

$$=\left[\frac{2}{mn}\{n^2+(m-1)(m-n)\}-(m-n-1)\frac{2}{mn}(m+n)\right]\cdot\frac{m-n}{2},$$

$$=\frac{m-n}{mn}\{n^2+(m-1)(m-n)-(m-n-1)(m+n)\},$$

$$=\frac{m-n}{mn}\{n^2+(m+n)n\},$$

$$=(m-n)\left\{1+\frac{2n}{m}\right\}.$$

Ex. 36. In an Arith$^{\circ}$. Prog^{n}. a is the first term, b the com. diff., and S_n the sum of n terms, prove that

$$S_n+S_{n+1}+S_{n+2}+\&\text{c. to } n \text{ terms}=(3n-1)\frac{na}{2}+(7n-2)(n-1)\frac{nb}{6}.$$

$$\begin{aligned}\text{Here } 2S_n &=\{2a+(n-1)b\}n,\\ 2S_{n+1}&=(2a+nb)(n+1),\\ 2S_{n+2}&=\{2a+(n+1)b\}(n+2),\\ \&\text{c.}&=\&\text{c.}\end{aligned}$$

$\therefore\ 2(S_n+S_{n+1}+S_{n+2}+\&\text{c. to } n \text{ terms})$

$$=\{n+(n+1)+(n+2)+\ldots\text{to } n \text{ terms}\}2a$$

$$+\{(n-1)n+n(n+1)+(n+1)(n+2)+\ldots\text{to } n \text{ terms}\}b,$$

$$=(3n-1)na+\{n^2-n+(n+1)^2-(n+1)+(n+2)^2-(n+2)+\&\text{c.}\}b,$$

$$=(3n-1)na+\{n^2+(n+1)^2+(n+2)^2+\ldots\text{to } n \text{ terms}$$

$$-(n+\overline{n+1}+\overline{n+2}+\ldots\text{to } n \text{ terms})\}b.$$

Now, (See Art. 295. Ex. 4. Cor.)

$$1^2+2^2+3^2+\ldots+(n-1)^2+n^2+(n+1)^2+\ldots\text{to } (2n-1) \text{ terms}$$

$$=\frac{1}{6}(2n-1)\,2n\,(4n-1).$$

And $1^2+2^2+3^2+\ldots+(n-1)^2=\frac{1}{6}(n-1)n(2n-1),$

$$\therefore\ n^2+(n+1)^2+(n+2)^2+\ldots\text{to } n \text{ terms}=\frac{1}{6}n(2n-1)(8n-2-n+1),$$

$$=\frac{1}{6}n(2n-1)(7n-1),$$

$$\therefore\ 2(S_n+S_{n+1}+\&c.)=(3n-1)na+\left\{\frac{1}{6}n(2n-1)(7n-1)-(3n-1)\frac{n}{2}\right\}b,$$

$$\therefore\ S_n+S_{n+1}+\&c.=(3n-1)\frac{na}{2}+\{(2n-1)(7n-1)-9n+3\}\frac{nb}{12},$$

$$=(3n-1)\frac{na}{2}+(2n-2)(7n-2)\frac{nb}{12},$$

$$=(3n-1)\frac{na}{2}+(n-1)(7n-2)\frac{nb}{6}.$$

Ex. 37. $S_1, S_2, S_3, \ldots S_p$ are the sums of p Arith^c. Prog^{ns}. to n terms; the first terms are 1, 2, 3, &c. and the common differences 1, 3, 5, 7, &c. Shew that $S_1+S_2+S_3+\ldots+S_p=(np+1)\frac{np}{2}$.

$$S_1=\{2+(n-1)1\}\frac{n}{2},$$

$$S_2=\{4+(n-1)3\}\frac{n}{2},$$

$$S_3=\{6+(n-1)5\}\frac{n}{2},$$

$$\&c.= \qquad \&c.$$

$$S_p=\{2p+(n-1)(2p-1)\}\frac{n}{2}.$$

$$\therefore\ S_1+S_2+S_3+\ldots+S_p$$

$$=\Big\{(2+4+6+\ldots+2p)+(n-1)[1+3+5+\ldots+(2p-1)]\Big\}\cdot\frac{n}{2},$$

$$=\left\{(2+2p)\frac{p}{2}+(n-1)[1+(2p-1)]\frac{p}{2}\right\}\cdot\frac{n}{2},$$

$$=(p+1+np-p)\frac{np}{2},$$

$$=(np+1)\frac{np}{2}.$$

Ex. 39. In the two series 2, 5, 8, &c., and 3, 7, 11, &c. each continued to 100 terms, find how many terms are identical.

The x^{th} term of the 1st series is $3x-1$,

... y^{th} .... 2nd .... $4y-1$.

Hence the x^{th} term of the one is identical with the y^{th} term of the other series, when $3x=4y$, that is, for all positive integral

values of x and y which satisfy this equation, *subject to the condition that neither* x *nor* y *exceeds* 100.

$$\text{Now} \quad 3x=4y, \quad \therefore\ x=y+\frac{y}{3},$$

and since x must be a whole number, $\therefore$ y must be divisible by 3 without remainder, i. e. $y=3q$, where q is a whole number, and then $x=4q$. So that putting 1, 2, 3, 4,.........25, for q, there are 25 values of x and y, and no more, which satisfy the equation $3x=4y$;

$\therefore$ the number of identical terms required is 25.

GEOMETRICAL AND HARMONICAL PROGRESSION.

Ex. 7. Sum $(s-a)+\{s-(a+ar)\}+\{s-(a+ar+ar^2)\}+\ldots$to n terms, and to ∞, where $s=a.\dfrac{r^n-1}{r-1}$.

(1) Sum to n terms

$$=ns-a.\frac{r-1}{r-1}-a.\frac{r^2-1}{r-1}-a.\frac{r^3-1}{r-1}-\&\text{c.}-a.\frac{r^n-1}{r-1},$$

$$=\frac{nar^n}{r-1}-\frac{na}{r-1}-\frac{ar}{r-1}(1+r+r^2+\ldots+r^{n-1})+\frac{na}{r-1},$$

$$=\frac{nar^n}{r-1}-\frac{ar}{r-1}.\frac{r^n-1}{r-1},$$

$$=\frac{nar^n}{r-1}-\frac{ar}{(r-1)^2}.(r^n-1).$$

(2) Sum to ∞ ....$=ns-\dfrac{ar}{r-1}(1+r+r^2+\ldots\text{to }\infty)+\dfrac{na}{r-1}$,

$$=-\frac{na}{r-1}-\frac{ar}{r-1}.\frac{1}{1-r}+\frac{na}{r-1},$$

$$=\frac{ar}{(1-r)^2}.$$

Ex. 17. In a Geom^e. Prog^n., if the $(p+q)^{\text{th}}$ term $=m$, and the $(p-q)^{\text{th}}$ term $=n$, shew that the p^{th} term $=\sqrt{mn}$, and the q^{th} term

$=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$. Also, if P be the p^{th} term, and Q the q^{th} term, shew that the n^{th} term $=\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

$$\text{Here } m = \text{the } (p+q)^{th} \text{ term} = ar^{p+q-1},$$

$$n = \ldots\ (p-q)^{th} \text{ term} = ar^{p-q-1},$$

$$\therefore\ mn = a^2r^{2p-2},$$

$$\therefore\ \sqrt{mn} = ar^{p-1} = \text{the } p^{th} \text{ term} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Again, } \frac{m}{n} = r^{2q},\quad \therefore\ r = \left(\frac{m}{n}\right)^{\frac{1}{2q}},\ \text{ and } \frac{1}{r^p} = \left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

$$\therefore\ \text{the } q^{th} \text{ term} = ar^{q-1} = \frac{m}{r^p} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\text{Also, } \left.\begin{array}{l} ar^{p-1} = P, \\ ar^{q-1} = Q, \end{array}\right\}\ \therefore\ \frac{ar^{p-1}}{ar^{q-1}} = \frac{P}{Q},\quad \therefore\ r^{p-q} = \frac{P}{Q},\quad r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}},$$

$$\text{and } a = \frac{P}{r^{p-1}} = P \div \left(\frac{P^{p-1}}{Q^{p-1}}\right)^{\frac{1}{p-q}},$$

$$\therefore\ \text{the } n^{th} \text{ term} = ar^{n-1} = P \div \left(\frac{P^{p-1}}{Q^{p-1}}\right)^{\frac{1}{p-q}} . \left(\frac{P^{n-1}}{Q^{n-1}}\right)^{\frac{1}{p-q}},$$

$$= \left(\frac{P^{p-q} . P^{-p+1} . P^{n-1}}{Q^{-p+1} . Q^{n-1}}\right)^{\frac{1}{p-q}} = \left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 18. Find the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th}, terms of a Geom^{c}. Series.

Let A be the first term of the Series, R the common Ratio; then

$$\left.\begin{array}{l} a = AR^{p-1}, \\ b = AR^{q-1}, \\ c = AR^{r-1}, \end{array}\right\}$$

$$\frac{a}{b} = R^{p-q},\quad \therefore\ R = \left(\frac{a}{b}\right)^{\frac{1}{p-q}}.$$

$$\text{Again, } \frac{b}{c} = R^{q-r} = \left(\frac{a^{q-r}}{b^{q-r}}\right)^{\frac{1}{p-q}},\quad \therefore\ \frac{b^{p-q}}{c^{p-q}} = \frac{a^{q-r}}{b^{q-r}},$$

$$\therefore\ b^{p-r} = a^{q-r}c^{p-q},\quad \text{or } a^{q-r}b^{r-p}c^{p-q} = 1.$$

Ex. 21. If a, b, c, d are quantities in G. P., prove that

$$a^2+b^2+c^2 > (a-b+c)^2;$$

and that $(a+b+c+d)^2 = (a+b)^2+(c+d)^2+2(b+c)^2$.

(1) $a^2+b^2+c^2-(a-b+c)^2=a^2+b^2+c^2-(a^2+b^2+c^2)+2ab+2bc-2ac,$

$$=2ab+2bc-2ac,$$

$$=2ab+2bc-2b^2, \text{ since } ac=b^2,$$

$$=2b(a+c-b).$$

But $(a+c)^2-b^2=a^2+c^2+2ac-b^2=a^2+c^2+b^2=$ a positive quantity,

$$\therefore (a+c)^2>b^2, \text{ and } a+c>b.$$

$$\therefore a^2+b^2+c^2-(a-b+c)^2=\text{a positive quantity,}$$

$$\text{or } a^2+b^2+c^2>(a-b+c)^2.$$

(2) $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(a+b)(c+d).$

But $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, $\therefore \dfrac{a+b}{b}=\dfrac{b+c}{c}=\dfrac{c+d}{d}$,

$$\therefore (a+b)(c+d)=\frac{bd}{c^2}(b+c)^2=(b+c)^2,$$

$$\therefore (a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2.$$

Ex. 22. If there be any number of quantities in G.P., r the common ratio, and S_m the sum of the first m terms, prove that the sum of the products of every two $=\dfrac{r}{r+1}.S_m.S_{m-1}$.

Let $a, b, c, \ldots h, k, l$ be m terms in G.P.

Then $S_m.S_{m-1}=(a+b+c+\ldots+k+l)(a+b+c+\ldots+k),$

$$=(a+b+c+\ldots+k)^2+l.S_{m-1},$$

$$=a^2+b^2+c^2+\ldots+k^2+2(ab+ac+bc+\ldots+hk)+l.S_{m-1},$$

$$\therefore ab+ac+bc+\ldots+hk=\frac{1}{2}S_m.S_{m-1}-\frac{1}{2}(a^2+b^2+c^2+\ldots+k^2)-\frac{l}{2}S_{m-1},$$

and the sum of the products of every two

$$=ab+ac+\ldots hk+(a+b+c+\ldots+h)l,$$

$\therefore$ sum of the products of every two

$$=\frac{1}{2}S_m.S_{m-1}-\frac{1}{2}(a^2+b^2+\ldots+k^2)+\frac{l}{2}.S_{m-1},$$

$$=\frac{1}{2}S_m.S_{m-1}-\frac{a^2}{2}.\frac{r^{2m-2}-1}{r^2-1}+\frac{ar^{m-1}}{2}.S_{m-1},$$

$$=\frac{1}{2}S_mS_{m-1}-\frac{a}{2}\cdot\frac{r^{m-1}+1}{r+1}\cdot a\frac{r^{m-1}-1}{r-1}+\frac{ar^{m-1}}{2}\cdot S_{m-1},$$

$$=\frac{1}{2}S_{m-1}\left\{S_m-a\cdot\frac{r^{m-1}+1}{r+1}+ar^{m-1}\right\},$$

$$=\frac{1}{2}S_{m-1}\left\{S_m+\frac{ar^m-a}{r+1}\right\},$$

$$=\frac{1}{2}S_{m-1}\left\{S_m+\frac{r-1}{r+1}\cdot S_m\right\},$$

$$=\frac{1}{2}S_{m-1}\cdot\frac{2r}{r+1}\cdot S_m,$$

$$=\frac{r}{r+1}\cdot S_m\cdot S_{m-1}.$$

Ex. 29. If a, b, c, be three terms in Harmonical Progression, shew that $a^2+c^2>2b^2$, if a and c have like signs.

Since a, b, c, are in H.P.

$$a : c :: a-b : b-c,$$

$$\therefore\ ab-ac=ac-bc,$$

$$2ac=ab+bc=(a+c)b,$$

$$\therefore\ b=\frac{2ac}{a+c}.$$

$$\therefore\ a^2+c^2-2b^2=a^2+c^2-2\left(\frac{2ac}{a+c}\right)^2,$$

$$=(a-c)^2+2ac\left\{1-\frac{4ac}{(a+c)^2}\right\},$$

$$=(a-c)^2+2ac\left(\frac{a-c}{a+c}\right)^2$$ = a positive quantity, if a and c have like signs,

$$\therefore\ a^2+c^2>2b^2.$$

Ex. 34. If between any two quantities there be inserted $(2n-1)$ Arith^c., Geom^c., and Harm^c., Means, shew that the n^{th} means are in G.P.

Let a, and b, be the two quantities; then, when the Means are inserted, there will be in each of the three series $\overline{2n+1}$ terms;

and the n^{th} mean in each case will be the $\overline{n+1}^{th}$, or *middle*, term of the series.

Now, in the Arith^c. Series the *middle* term will be the Arith^c. Mean between the first and last terms; and therefore, in this case, the n^{th} Mean $=\dfrac{a+b}{2}$.

In the Geom^c. Series the *middle* term will, likewise, be the Geom^c. Mean between the first and last terms; and therefore, in this case,

$$\text{the } n^{th} \text{ Mean} = \sqrt{ab}.$$

In the Harm^c. Series the reciprocal of the *middle* term will be the Arith^c. Mean between the reciprocals of the first and last terms; and therefore the reciprocal of the n^{th} Mean

$$=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right),$$

$$=\frac{a+b}{2ab};$$

$$\therefore \text{ the } n^{th} \text{ Harm}^c\text{. Mean} = \frac{2ab}{a+b}.$$

Hence the n^{th} Means are $\dfrac{a+b}{2}$, $\sqrt{ab}$, and $\dfrac{2ab}{a+b}$,

and Prod^t. of the extremes = the square of the means,

$\therefore$ the 3 quantities are in G.P.

Ex. 36. If a_1, a_2, a_3 are in A.P., (1), a_2, a_3, a_4 in G.P., (2), and a_3, a_4, a_5 in H.P., (3), shew that a_1, a_3, a_5 are in G.P.

$$a_3^2 = a_2 . a_4, \text{ from (2)},$$

$$=\frac{a_1+a_3}{2}.a_4, \text{ from (1)},$$

$$=\frac{a_1+a_3}{2}.\frac{2a_3a_5}{a_3+a_5}, \text{ from (3)},$$

$$=\frac{a_1+a_3}{a_3+a_5}.a_3a_5,$$

$$\therefore\ a_3 = \frac{a_1+a_3}{a_3+a_5} \cdot a_5,$$

$$a_3^2 + a_3 a_5 = a_1 a_5 + a_3 a_5,$$

$$a_3^2 = a_1 a_5,$$

$$\therefore\ a_1,\ a_3,\ a_5 \text{ are in G.P.}$$

Ex. 37. Find the relation between a, b, c, that they may be the p^{th}, q^{th}, and r^{th}, terms of an Harm^c. Prog^n.

If a, b, c, are the p^{th}, q^{th}, r^{th} terms of an H.P.

then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$,............................. A.P. (Art. 293)

$\therefore$ substituting $\frac{1}{a}$ for a, $\frac{1}{b}$ for b, and $\frac{1}{c}$ for c, in Ex. 32. p. 97, we have

$$(q-r)\frac{1}{a} + (r-p)\frac{1}{b} + (p-q)\frac{1}{c} = 0,$$

$$\therefore\ (q-r)bc + (r-p)ac + (p-q)ab = 0.$$

Ex. 38. If a, b, c, be in G.P., shew that $\log_a n$, $\log_b n$, $\log_c n$, are in H.P.

$$\text{Let } x = \log_a n, \quad \therefore\ n = a^x,$$

$$y = \log_b n, \qquad n = b^y,$$

$$z = \log_c n, \qquad n = c^z,$$

$$\therefore\ a^x = b^y = c^z.$$

$$\therefore\ (ac)^x = c^{x+z}, \quad \text{or } b^{2x} = c^{x+z}, \text{ since } b^2 = ac.$$

$$\text{But } b = c^{\frac{z}{y}}, \quad \therefore\ b^{2x} = c^{\frac{2xz}{y}},$$

$$\therefore\ c^{\frac{2xz}{y}} = c^{x+z},$$

$$\therefore\ \frac{2xz}{y} = x+z, \quad \text{or } y = \frac{2xz}{x+z}.$$

Hence y is the Harmonic Mean between x and z, (Art. 294, Cor.) that is, x, y, z are in Harmonic Progression.

Ex. 39. Find the sum of $2p$ terms of the series whose n^{th} term is $n\{[(-1)^n+1]n+2\}$.

The 1st term $=1\times2$,

2nd ... $=2(2+2\times2)$,

3rd ... $=3\times2$,

4th ... $=4(2+4\times2)$,

$2p^{\text{th}}$ ... $=2p(2+2p\times2)$,

$\therefore$ sum of the series $=2(1+2+3+\ldots+2p)+2(2^2+4^2+\&c.+\overline{2p}|^2)$,

$$=2(2p+1)p+8(1^2+2^2+3^2+\ldots+p^2),$$

$$=2p(2p+1)+\frac{4}{3}p(p+1)(2p+1),$$

$$=2p(2p+1)\{1+\frac{2}{3}(p+1)\},$$

$$=\frac{2p}{3}(2p+1)(2p+5).$$

PERMUTATIONS AND COMBINATIONS.

Ex. 5. How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence?

There are 6 different coins, and they may be combined any number together to make a sum.

Taken 6 together, number of different sums .... $=1$,

...... 5 ...., ... $=6$,

...... 4 ...., ... $=\frac{6\times5}{1\times2}=15$,

...... 3 ...., ... $=\frac{6\times5\times4}{1\times2\times3}=20$,

...... 2 ...., ... $=\frac{6\times5}{1\times2}=15$,

...... 1 ...., ... $=6$,

$\therefore$ number required $=1+6+15+20+15+6=63$.

Ex. 6. In the Permutations formed out of a, b, c, d, e, f, g, taken all together, how many begin with ab? How many with abc? How many with $abcd$?

(1) Number of Permutations beginning with ab = number of Permutations of c, d, e, f, g; since it is only by the different arrangement of c, d, e, f, g, that the Permutations in question are formed,

$\therefore$ number required $= 5\times4\times3\times2\times1 = 120$.

(2) Similarly, number of Permutations beginning with abc = number of Permutations of $d, e, f, g = 4\times3\times2\times1 = 24$.

(3) And, number of Permutations beginning with $abcd$ = $3\times2\times1 = 6$.

Ex. 7. Of the combinations of 10 letters, a, b, c, &c. taken 5 together, in how many will a occur?

To make a combination of 5 letters, which shall include a, it is evident that there must be 4 other letters to which a is added, and these may be *any* 4 out of the 9 letters, b, c, d, &c.

$\therefore$ Number required = number of combinations of 9 things taken 4 together,

$$= \frac{9\times8\times7\times6}{1\times2\times3\times4} = 126.$$

Ex. 9. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates, and 3 members to be chosen; in how many ways may a man vote?

He may vote for 1 candidate only, or for any 2, or for any 3;

Number of ways of voting for 1 = 4,

.... 2 = No. of combs. of 4 things taken 2 together.

$$= \frac{4\times3}{1\times2} = 6,$$

.... $3 = \frac{4\times3\times2}{1\times2\times3} = 4$,

$\therefore$ Total number of ways $= 4+6+4 = 14$.

Ex. 12. Find the number of different triangles into which a polygon of n sides may be divided by joining the angular points.

Since there are n sides, there are also n angular points, of which *three* must go to form a triangle, but they may be *any*

three. Hence the number of such triangles = number of combinations of n things taken 3 together,

$$=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{1}{6}n(n-1)(n-2).$$

Ex. 16. If there be any unknown number of beans in a bag, prove that the chance of bringing out an odd number taken at random is greater than that of bringing out an even number, excluding the case of bringing out none at all.

Since $(1-x)^n=1-C_1x+C_2x^2-C_3x^3+C_4x^4-C_5x^5+C_6x^6-\&c.$

where C_1 = number of combinations of n things taken 1 together,

C_2 = ... 2 ......

C_3 = ... 3 ......;

and so on;

$$\therefore \text{ if } x=1,\quad 1-C_1+C_2-C_3+C_4-C_5+C_6-\&c.=0,$$

$$\therefore\ C_1+C_3+C_5+\&c.=1+C_2+C_4+C_6+\&c.$$

$$\text{or }\ C_1+C_3+C_5+\&c.>C_2+C_4+C_6+\&c.$$

or, total number of combinations with an odd number together > total number of combinations with an even number together.

i. e. chance of bringing out an odd number > chance of bringing out an even number.

Ex. 17. In how many ways can 8 persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements?

It is obvious, that all the different arrangements of 7 of the persons, while the 8^{th} remains fixed, will be those in which all the 8 persons have not the same neighbours, at least in the same position, and the number of these arrangements

$$=7\times6\times5\times4\times3\times2\times1=5040.$$

But as one half of these arrangements will be similar to the other half, if the *position* of neighbours on the right hand or the left be not regarded as making a difference, i. e. if $B, A, C,$ makes A have the *same* neighbours, as $C, A, B,$ then the correct answer to the question will be

$$\frac{1}{2}\times5040, \text{ or } 2520.$$

Ex. 18. Out of 17 consonants and 5 vowels, how many words can be made having 2 consonants and 1 vowel in each?

Number of different arrangements of 17 consonants 2 together $=17\times16$;

and with *each* of these, 3 different arrangements may be made by means of *one* vowel,

$\therefore$ number of words, having given only *one* vowel $=17\times16\times3$,

and the same is true for *each* of the 5 different vowels,

$\therefore$ number of words required $=17\times16\times3\times5$,

$=4080.$

Ex. 20. Find the number of combinations that can be formed out of the letters of the word '*Notation*' taken 3 together.

Here are 5 *different* letters, $\therefore$ the number of combinations of 5 letters, 3 together, where no letter recurs, $=\frac{5\times4}{1\times2}=10.$

Also there are 2 n's, 2 o's, and 2 t's, *each* of which pairs may be combined with each of the other 4 letters, and form 4 combinations of 3, making altogether 3×4, or 12, such combinations, where the letters recur,

$\therefore$ number required $=10+12=22.$

Ex. 21. If there be 2 dice, one of which has n, and the other $(n+r)$, faces, each die being marked in the usual manner from 1 upwards, find the number of different throws which can be made with them.

Here the dice are supposed always to be thrown *together;* and if all the faces of both were marked *differently*, then since each face of the one die combined with each of the other would form a *different* combination, it is obvious that the number of different throws would be

$$n(n+r).$$

But this number is diminished in the present case, because the two dice have n faces *alike*, being marked,

$$1,\ 2,\ 3\ldots\ldots n,$$

$$1,\ 2,\ 3\ldots\ldots n,\ n+1,\ \ldots\ldots n+r,$$

so that among the throws which have been assumed to be *different* in the first estimate are [1, 2], [2, 1], which are really the same; [1, 3], [3, 1]; and so on. Therefore we must deduct the number

of combinations of n things taken 2 together; and the number of *different* throws

$$=n(n+r)-\frac{1}{2}n(n-1),$$

$$=\frac{1}{2}n(n+1)+nr.$$

Ex. 23. Find the total number of different combinations of n things taken 1, 2, 3, ...n together, of which there are p of one sort, q of another, r of another, &c.

Let a, b, c, &c. be the different given things, of which a recurs p times, b recurs q times, c recurs r times, &c. making n things in all, so that being represented by $a^p b^q c^r$, &c. it is obvious that the number of divisors of this quantity will be the total number of combinations required. Now the number of divisors of $a^p b^q c^r$, &c. is found in Art. 378 to be

$$(p+1)(q+1)(r+1), \text{ \&c.}$$

"including 1 and $a^p b^q c^r$, &c. itself." In the present case the latter is to be included but not the former,

$\therefore$ number of combinations required $=(p+1)(q+1)(r+1)$ &c. -1.

BINOMIAL THEOREM, ETC.

Ex. 24. Find an approximate value of $\sqrt[7]{108}$.

Since $1^7=1$, and $2^7=128$, $\therefore \sqrt[7]{108}$ is nearest to 2;

$$\sqrt[7]{108}=\sqrt[7]{2^7-20}=2\left\{1-\frac{20}{2^7}\right\}^{\frac{1}{7}}=2\left\{1-\frac{5}{2^5}\right\}^{\frac{1}{7}}=2\left\{1-\frac{5^6}{10^5}\right\}^{\frac{1}{7}},$$

$$=2\left\{1-\frac{1}{7}\times\frac{5^6}{10^5}-\frac{1}{7}\times\frac{3}{7}\times\left(\frac{5^6}{10^5}\right)^2-\frac{1}{7}\times\frac{3}{7}\times\frac{13}{3\times7}\times\left(\frac{5^6}{10^5}\right)^3-\ldots\ldots\right\}.$$

$$\text{Now } \frac{1}{7}\times\frac{5^6}{10^5}=\frac{1}{7}\times\frac{15625}{10^5}=\frac{0\cdot15625}{7}=0\cdot022321,$$

$$\frac{1}{7}\times\frac{3}{7}\left(\frac{5^6}{10^5}\right)^2=3\times(0\cdot022321)^2=3\times0\cdot000498=0\cdot001494,$$

$$\frac{1}{7}\times\frac{3}{7}\times\frac{13}{3\times7}\left(\frac{5^6}{10^5}\right)^3=0\cdot001494\times\frac{13}{3\times7}\times0\cdot15625=0\cdot000144,$$

$$\therefore \sqrt[7]{108} = 2\times\{1-0\text{·}023959\} = 2\times 0\text{·}976041,$$
$$= 1\text{·}9520, \text{ nearly.}$$

Ex. 25. Find the sum of $1+\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{4}+\&c.$, $1, \alpha, \beta, \gamma$, &c. being the coefficients taken in order of the expansion of $(a+b)^n$.

$$(n+1)(1+\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{4}+\&c.)$$
$$=(n+1)\{1+\frac{n}{1\times 2}+\frac{n(n-1)}{1\times 2\times 3}+\frac{n(n-1)(n-2)}{1\times 2\times 3\times 4}+\&c.$$
$$=-1+1+(n+1)+\frac{(n+1)n}{1\times 2}+\frac{(n+1)n(n-1)}{1\times 2\times 3}+\&c.$$
$$=-1+(1+1)^{n+1}, \text{ Art. } 320,$$
$$=-1+2^{n+1},$$
$$\therefore 1+\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{4}+\&c. = \frac{2^{n+1}-1}{n+1}.$$

Ex. 27. If a, b, c, d be the 6^{th}, 7^{th}, 8^{th}, 9^{th} terms respectively of an expanded binomial, shew that

$$\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}.$$

Let $x+y$ be the binomial, then by the law of expansion we know that

$$\left.\begin{aligned} \text{The } 7^{th} \text{ term, } b &= \frac{a}{6}.\frac{y}{x}, \ldots(1) \\ \text{also } c &= \frac{b}{7}.\frac{y}{x}, \ldots(2) \\ \text{and } d &= \frac{c}{8}.\frac{y}{x}, \ldots(3) \end{aligned}\right\}$$

from (1) and (2), $\frac{b}{c}=\frac{7a}{6b}$, $\therefore b^2=\frac{7ac}{6}$, and $b^2-ac=\frac{ac}{6}$,

from (2) and (3), $\frac{c}{d}=\frac{8b}{7c}$, $\therefore c^2=\frac{8bd}{7}$, and $c^2-bd=\frac{bd}{7}$,

from (1) and (3), $\frac{b}{d}=\frac{4a}{3c}$,

$$\therefore \frac{b^2-ac}{c^2-bd}=\frac{7ac}{6bd}=\frac{b}{d}=\frac{4a}{3c}.$$

Ex. 28. Find the coefficient of x^n in the expansion of $\frac{(1-2x)^2}{(1-x)^4}$,

$$\frac{(1-2x)^2}{(1-x)^4}=(1-4x+4x^2)(1+4x+\frac{4\times5}{1\times2}x^2+\frac{4\times5\times6}{1\times2\times3}x^3+\ldots\ldots);$$

taking from the latter factor the 3 terms involving x^n, x^{n-1}, and x^{n-2},

$$\text{coefficient required}=1\times\frac{4\times5\times6\ldots(n+3)}{1\times2\times3\ldots n}-4\times\frac{4\times5\times6\ldots(n+2)}{1\times2\times3\ldots(n-1)}$$
$$+4\times\frac{4\times5\times6\ldots(n+1)}{1\times2\times3\ldots(n-2)}.$$

Now

$$\frac{4\times5\times6\ldots(n+3)}{1\times2\times3\ldots\ n}=\frac{4\times5\times6\ldots n(n+1)(n+2)(n+3)}{4\times5\times6\ldots n\,.\,1\times2\times3}=\frac{(n+1)(n+2)(n+3)}{1\times2\times3}.$$

$\therefore$ coefficient required

$$=\frac{(n+1)(n+2)(n+3)}{1\times2\times3}-4\frac{n(n+1)(n+2)}{1\times2\times3}+4\frac{(n-1)n(n+1)}{1\times2\times3},$$

$$=\frac{1}{6}(n+1)\{(n+2)(n+3)-4n(n+2)+4n(n-1)\},$$

$$=\frac{1}{6}(n+1)(n^2+5n+6-12n),$$

$$=\frac{1}{6}(n+1)(n^2-7n+6),$$

$$=\frac{1}{6}(n+1)(n-1)(n-6),$$

$$=\frac{1}{6}(n^2-1)(n-6).$$

Ex. 29. If r be the greatest whole number contained in $\frac{p}{q}$, $(a+x)^{\frac{p}{q}}$ has the first $(r+2)$ terms of its expansion positive, and the $(r+m)^{\text{th}}$ of the same sign as $(-1)^m$; but $(a-x)^{\frac{p}{q}}$ has the first $(r+1)$ terms alternately positive and negative, and all the rest of the same sign as $(-1)^{r+1}$.

Let $\frac{p}{q}=r+f$, where f is a fraction, that is, <1;

$$\text{then } (a+x)^{\frac{p}{q}}=(a+x)^{r+f},$$

of which expanded the coefficient of the $(s+1)^{\text{th}}$ term is

$$\frac{(r+f)(r+f-1)\ldots\ldots(r+f-s+1)}{1 \times 2 \quad \ldots\ldots\ldots\ldots s},$$

and this will be positive as long as s is not greater than $r+f+1$, or $r+1$, f being less than 1, that is, until $s+1=r+2$, that case included;

$\therefore$ the first $(r+2)$ terms are positive.

After that the terms will be negative or positive alternately, according as there are an odd or even number of negative factors in the coefficient,

the $(s+2)^{\text{th}}$ or $(r+3)^{\text{th}}$ having the same sign as -1, or $(-1)^3$,

$(r+4)^{\text{th}}$ .... $(-1)^4$

....

$(r+m)^{\text{th}}$ .... $(-1)^m$.

2ndly. The $(s+1)^{\text{th}}$ *term* of $(a-x)^{\frac{p}{q}}$ is

$$\frac{(r+f)(r+f-1)\ldots\ldots(r+f-s+1)}{1 \times 2 \quad \ldots\ldots\ldots\ldots s} . a^{\frac{p}{q}-s}(-x)^s.$$

Now from above it appears that for all values of s, until $s=r$, inclusive, that is, for the first $(r+1)$ terms, the *coefficient* is positive, and $(-x)^s$ will evidently be alternately positive and negative; therefore the *terms* are alternately positive and negative. But afterwards, if $(r+1)$ be even, $(-x)^{r+1}$ will be positive, and the coefficient positive, $\therefore$ the *term* is positive. And the next term will have one negative factor in the coefficient, and $(-x)^{r+2}$ will be negative; $\therefore$ that term will be positive; and so on: i.e. all the terms after the first $(r+1)$ will be positive, or of the same sign as $(-1)^{r+1}$. Again, if $(r+1)$ be odd, $(-1)^{r+1}$ is negative, and the coefficient is positive, but $(-x)^{r+1}$ is negative, $\therefore$ the term will be negative: and the next term will have a negative coefficient $\times(-x)^{r+2}$, and $\therefore$ will be negative; and so on for the succeeding terms; that is, all the terms after the first $(r+1)$ will be of the same sign as $(-1)^{r+1}$.

Ex. 30. Given that the coefficient of the $(p+1)^{\text{th}}$ term of the expansion of $(1+x)^n$ is equal to that of the $(p+3)^{\text{th}}$ term, shew that $p=\frac{n}{2}-1$.

By the question,

$$\frac{n(n-1)\ldots(n-p+1)}{1 \times 2 \ldots\ldots p} = \frac{n(n-1)\ldots(n-p-1)}{1\times 2\ldots\ldots(p+2)},$$

$$= \frac{n(n-1)\ldots(n-p+1)(n-p)(n-p-1)}{1 \times 2 \ldots\ldots\ldots p \quad (p+1)(p+2)},$$

$$\therefore\ 1=\frac{(n-p)(n-p-1)}{(p+1)(p+2)},$$

$$(p+1)(p+2)=(n-p)^2-n+p,$$

$$p^2+3p+2=n^2-2np+p^2-n+p,$$

$$2p(n+1)=n^2-n-2=(n+1)(n-2),$$

$$\therefore\ 2p=n-2,$$

$$\therefore\ p=\frac{n}{2}-1.$$

Ex. 32. If $x>a$, prove that the sum of all the terms of the expansion of $(x+a)^n$, after the first two, is less than $(2^n-n-1)ax^{n-1}$.

The sum of the terms after the first two are

$$\frac{n}{1}\cdot\frac{n-1}{2}a^2x^{n-2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^3x^{n-3}+\&c.,$$

$$\text{or }\left\{\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{a}{x}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{a^2}{x^2}+\&c.\right\}ax^{n-1}.$$

But, by supposition, each of the quantities $\frac{a}{x}$, $\frac{x^2}{x^2}$, &c. is less than 1; therefore this sum is *less than*

$$\left\{\frac{n}{1}\cdot\frac{n-1}{2}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}+\&c.\right\}ax^{n-1},$$

$$\text{or }\{(1+1)^n-1-n\}ax^{n-1},$$

$$\text{or }(2^n-n-1)ax^{n-1}.$$

Ex. 33. If $\frac{p(p-1)(p-2)\ldots(p-r+1)}{1\times2\times3\ \ldots\ldots r}$ be represented by p_r, prove that

$$(p+q)_r=p_r+p_{r-1}q_1+p_{r-2}q_2+\ldots+p_1q_{r-1}+q_r.$$

Since p_r is the coefficient of the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^p$,

$$(1+x)^p=1+p_1x+p_2x^2+p_3x^3+\ldots+p_{r-1}x^{r-1}+p_rx^r+\ldots$$

$$\text{and }(1+x)^q=1+q_1x+q_2x^2+q_3x^3+\ldots+q_{r-1}x^{r-1}+q_rx^r+\ldots$$

$\therefore\ (1+x)^{p+q}=$ the product of these two series..............(1)

But $(1+x)^{p+q}=1+(p+q)_1x+(p+q)_2x^2+\ldots+(p+q)_rx^r+\ldots\ldots$ (2)

and since (1) and (2) are identically equal, $\therefore$ the coefficients of the same powers of x are equal, as of x^r,

$$\therefore (p+q)_r=p_r+p_{r-1}q_1+p_{r-2}q_2+\ldots+p_1q_{r-1}+q_r.$$

Ex. 34. If $\dfrac{n(n+1)(n+2)\ldots(n+r-1)}{1\times 2\times 3\ \ldots\ldots r}$ be represented by ${}_nP_r$, shew that

$${}_nP_r+{}_{n+1}P_{r-1}={}_{n+1}P_r.$$

Writing $n+1$ for n in the given quantity,

$${}_{n+1}P_r=\frac{(n+1)(n+2)\ldots(n+r)}{1\times 2\ldots\ldots\ldots r},$$

$$\text{and } {}_{n+1}P_{r-1}=\frac{(n+1)(n+2)\ldots(n+r-1)}{1\times 2\ldots\ldots(r-1)}.$$

$$\therefore\ {}_nP_r+{}_{n+1}P_{r-1}=\frac{n(n+1)\ldots(n+r-1)}{1\times 2\ldots\ldots\ldots r}+\frac{(n+1)(n+2)\ldots(n+r-1)}{1\times 2\ldots\ldots(r-1)},$$

$$=\frac{(n+1)(n+2)\ldots(n+r-1)}{1\times 2\ldots\ldots\ldots(r-1)}\cdot\left(\frac{n}{r}+1\right),$$

$$=\frac{(n+1)(n+2)\ldots(n+r-1)(n+r)}{1\times 2\ldots\ldots\ldots(r-1)r},$$

$$={}_{n+1}P_r.$$

Ex. 35. Prove that

$$\frac{1.3.5\ldots(2r-1)}{\lfloor r}+\frac{1.3.5\ldots(2r-3)}{\lfloor r-1}\cdot\frac{3}{\lfloor 1}+\frac{1.3.5\ldots(2r-5)}{\lfloor r-2}\cdot\frac{3\times 5}{\lfloor 2}+\&c.=2^r(1+r).$$

$$(1-x)^{-\frac{1}{2}}=1+\frac{x}{2}+\frac{1\times 3}{2^2}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2^3}\cdot\frac{x^3}{\lfloor 3}+\ldots+\frac{1.3.5\ldots(2r-1)}{2^r}\cdot\frac{x^r}{\lfloor r}+\ldots\ldots(A)$$

$$(1-x)^{-\frac{3}{2}}=1+\frac{3x}{2}+\frac{3\times 5}{2^2}\cdot\frac{x^2}{\lfloor 2}+\ldots\ldots\ldots\ldots(B)$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots\ldots\ldots\ldots(C)$$

Now (C) is the product of (A) and (B), $\therefore$ equating coefficient of x^r in this product with that in (C), we have

$$r+1=\frac{1.3.5\ldots(2r-1)}{2^r}\cdot\frac{1}{\underline{|r}}+\frac{1.3.5\ldots(2r-3)}{2^{r-1}}\cdot\frac{1}{\underline{|r-1}}\cdot\frac{3}{2}$$

$$+\frac{1.3.5\ldots(2r-5)}{2^{r-2}}\cdot\frac{1}{\underline{|r-2}}\cdot\frac{3\times5}{2^2}\cdot\frac{1}{\underline{|2}}+\&c.$$

$$\therefore\ \frac{1.3.5\ldots(2r-1)}{\underline{|r}}+\frac{1.3.5\ldots(2r-3)}{\underline{|r-1}}\cdot\frac{3}{\underline{|1}}+\frac{1.3.5\ldots(2r-5)}{\underline{|r-2}}\cdot\frac{3\times5}{\underline{|2}}+\&c.$$

$$=2^r(1+r).$$

Ex. 36. Shew that if t_r denote the middle term of $(1+x)^{2r}$, then will

$$t_0+t_1+t_2+\ldots\ldots=(1-4x)^{-\frac{1}{2}}.$$

$t_r=$ the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^{2r}$,

$$=\frac{2r(2r-1)(2r-2)\ldots(r+1)}{1\ .\ 2\ .\ 3\ldots\ldots\ldots r}.x^r,$$

$$=\frac{1.2.3\ldots r(r+1)\ldots(2r-1)2r}{(1.2.3\ldots r)^2}.x^r,$$

$$=\frac{1.3.5\ldots(2r-1).2.4.6\ldots 2r}{(1.2.3\ldots r)^2}.x^r,$$

$$=\frac{1.3.5\ldots(2r-1).1.2.3\ldots r.2^r}{(1.2.3\ldots r)^2}.x^r,$$

$$=\frac{1.3.5\ldots(2r-1)}{1.2.3\ldots\ldots r}(2x)^r.$$

$$\left.\begin{array}{r}\text{Also the }(r+1)^{\text{th}}\text{ term}\\ \text{of }(1-4x)^{-\frac{1}{2}}\end{array}\right\}=\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots-\frac{2r-1}{2}}{1\ .\ 2\ .\ 3\ldots\ldots\ldots r}(-4x)^r,$$

$$=\left(-\frac{1}{2}\right)^r\cdot\frac{1.3.5\ldots(2r-1)}{1.2.3\ldots\ldots r}(-2)^r(2x)^r,$$

$$=\frac{1.3.5\ldots(2r-1)}{1.2.3\ldots\ldots r}(2x)^r=t_r,$$

and since all the terms of the expansion of $(1-4x)^{-\frac{1}{2}}$ are found by substituting 0, 1, 2, 3, &c. successively for r in the general, or $(r+1)^{\text{th}}$, term,

$$\therefore\ t_0+t_1+t_2+\ldots in\ inf.=(1-4x)^{-\frac{1}{2}}.$$

Ex. 37. If x be very small compared with 1, prove that

$$\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}=1-\frac{5}{6}x, \text{ nearly.}$$

Given fraction $=\dfrac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{1+x+(1+x)^{\frac{1}{2}}}$,

$=\dfrac{1+\frac{1}{2}x+1-\frac{2}{3}x}{1+x+1+\frac{1}{2}x}$ nearly, expanding and neglecting the higher powers of x as being very small.

$$=\frac{2-\frac{1}{6}x}{2+\frac{3}{2}x}=\frac{1-\frac{1}{12}x}{1+\frac{3}{4}x},$$

$$=\left(1-\frac{1}{12}x\right)\left(1+\frac{3}{4}x\right)^{-1},$$

$$=\left(1-\frac{1}{12}x\right)\left(1-\frac{3}{4}x\right), \text{ nearly,}$$

$$=1-\frac{1}{12}x-\frac{3}{4}x, \quad \ldots\ldots$$

$$=1-\frac{5}{6}x, \quad \ldots\ldots$$

Ex. 38. If $c=a-b$, and is very small compared with a and b, shew that $\dfrac{a^2b^2}{(a^2-a^2x^2+b^2x^2)^{\frac{3}{2}}}=a-2c+3cx^2$, nearly.

Given fraction $=a^2b^2.\{a^2-(a^2-b^2)x^2\}^{-\frac{3}{2}}$,

$$=a^2b^2.\{a^2(1-\frac{a^2-b^2}{a^2}.x^2)\}^{-\frac{3}{2}},$$

$$=\frac{b^2}{a}\{1-\frac{a^2-b^2}{a^2}.x^2\}^{-\frac{3}{2}}.$$

Now $b=a-c$, $\therefore\ b^2=a^2-2ac$, neglecting c^2 as being very small,

$$\therefore\ \frac{b^2}{a}=a-2c, \text{ and } \frac{a^2-b^2}{a^2}=\frac{2c}{a},$$

$\therefore$ Given fraction $=(a-2c)(1-\frac{2c}{a}x^2)^{-\frac{3}{2}}$,

$=(a-2c)(1+\frac{3c}{a}x^2)$, nearly,

$=a-2c+3cx^2$, ...

Ex. 39. If x be nearly equal to 1, shew that $\frac{mx^m-nx^n}{m-n}=x^{m+n}$ nearly.

Let $x=1+h$, h being very small; then

$$\frac{mx^m-nx^n}{m-n}=\frac{m(1+h)^m-n(1+h)^n}{m-n}=\frac{m(1+mh)-n(1+nh)}{m-n}, \text{ nearly,}$$

$$=\frac{m-n+(m^2-n^2)h}{m-n}=1+(m+n)h,$$

$=(1+h)^{m+n}$, nearly,

$=x^{m+n}$, ...

Ex. 41. If $u=\frac{a^p}{z^m}+\frac{b^q-a^p}{z^n}$, and z is very nearly equal to 1, shew that $u=b^q.z^{(n-m)\frac{a^p}{b^q}-n}$, very nearly.

Let $z=1+h$, where h is very small; then

$u=a^p(1+h)^{-m}+(b^q-a^p)(1+h)^{-n}$,

$=a^p(1-mh)+(b^q-a^p)(1-nh)$, very nearly,

$=a^p-ma^ph+b^q-a^p-nb^qh+na^ph$, ...

$=b^q.\{1+(\overline{n-m}.\frac{a^p}{b^q}-n)h\}$, ...

$=b^q(1+h)^{(n-m)\frac{a^p}{b^q}-n}$, ...

$=b^q.z^{(n-m)\frac{a^p}{b^q}-n}$, very nearly.

Ex. 42. If $\frac{x}{a}=1+h$, h being very small, find the value of

$$\frac{\left(2ax-x^2+\frac{b^2x^2}{a^2}\right)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}.$$

Given fraction $=\dfrac{1}{(a^2+b^2)^{\frac{1}{2}}}\cdot\left\{2a^2.\dfrac{x}{a}-(a^2-b^2)\dfrac{x^2}{a^2}\right\}^{\frac{1}{2}}$,

$$=\frac{1}{(a^2+b^2)^{\frac{1}{2}}}\cdot\left\{2a^2(1+h)-(a^2-b^2)(1+2h)\right\}^{\frac{1}{2}},\text{ nearly,}$$

$$=\frac{1}{(a^2+b^2)^{\frac{1}{2}}}\cdot\left\{a^2+b^2+2b^2h\right\}^{\frac{1}{2}},\qquad \ldots\ldots\ldots$$

$$=\left(1+\frac{2b^2h}{a^2+b^2}\right)^{\frac{1}{2}},\qquad \ldots\ldots\ldots$$

$$=1+\frac{b^2h}{a^2+b^2},\text{ nearly.}$$

Ex. 43. If N represent the n^{th} term of the expansion of a^x, find n when the series begins to converge at that term; and shew that the sum of all the succeeding terms is less than $\dfrac{Nn}{n-x.\log_e a}$.

(1) By Art. 325, $a^x=1+\dfrac{Ax}{1}+\dfrac{A^2x^2}{1.2}+\dfrac{A^3x^3}{1.2.3}+\ldots\ldots$ where $A=\log_e a$.

$$\therefore\ \frac{n^{\text{th}}\text{ term}}{(n-1)^{\text{th}}\text{ term}}=\frac{A^{n-1}x^{n-1}}{\lfloor n-1}\div\frac{A^{n-2}x^{n-2}}{\lfloor n-2}=\frac{Ax}{n-1},$$

$$\therefore\ n^{\text{th}}\text{ term}=\frac{Ax}{n-1}\times(n-1)^{\text{th}}\text{ term},$$

and the series will begin to converge at the n^{th} term, when n is such, that $\dfrac{Ax}{n-1}$ is first less than 1, that is, when $n-1$ is first greater than Ax,

$\therefore$ n is the first whole number greater than $1+Ax$, or $1+x\log_e a$.

(2) The sum of the remaining terms after the n^{th}

$$=\frac{A^nx^n}{\lfloor n}+\frac{A^{n+1}x^{n+1}}{\lfloor n+1}+\ldots\ldots$$

$$=\frac{NAx}{n}\cdot\left\{1+\frac{Ax}{n+1}+\frac{A^2x^2}{(n+1)(n+2)}+\ldots\ldots\right\},$$

$$<\frac{NAx}{n}\cdot\left\{1+\frac{Ax}{n}+\frac{A^2x^2}{n^2}+\frac{A^3x^3}{n^3}+\ldots\ldots\ldots\right\},$$

$$<\frac{NAx}{n}\cdot\left\{1-\frac{Ax}{n}\right\}^{-1},$$

$$< \frac{NAx}{n-Ax},$$

$$< \frac{Nn}{n-Ax}, \text{ since } n > Ax,$$

$$< \frac{Nn}{n-x.\log_e a}.$$

Ex. 48. Find the coefficient of x^4 in $(2+\sqrt{x}-x)^5$.

For the given multinomial write $(a+bx^{\frac{1}{2}}+cx)^5$, or $(a+by+cy^2)^5$, then the coefficient required will be that of y^8, since $x^4=y^8$.

Here $\left.\begin{array}{r} p+q+r=5, \\ q+2r=8, \end{array}\right\}$ are the equations of condition.

p	q	r
1	0	4
0	2	3

$$\therefore \text{ coefficient required} = \lfloor 5 . \left\{\frac{ac^4}{\lfloor 4} + \frac{b^2c^3}{\lfloor 2 . \lfloor 3}\right\},$$

$$= \lfloor 5 . \left\{\frac{2\times(-1)^4}{\lfloor 4} + \frac{1^2\times(-1)^3}{\lfloor 2 . \lfloor 3}\right\},$$

$$= 5\times\{2-2\} = 0.$$

Ex. 49. Find the terms which involve $\frac{yz^7}{x^2}$, and y^5z, in $\left(\frac{x^2}{y}-\frac{y^2}{z}+\frac{z^2}{x}\right)^6$.

Here $\frac{yz^7}{x^2} = \frac{x^2}{y}.\frac{y^2}{z}.\left(\frac{z^2}{x}\right)^4$, and $y^5z = \frac{x^2}{y}.\left(\frac{y^2}{z}\right)^3.\left(\frac{z^2}{x}\right)^2$.

[If this part is not readily done by *inspection*, it may be done as follows:

(1) Let $\frac{yz^7}{x^2} = \left(\frac{x^2}{y}\right)^m\left(\frac{y^2}{z}\right)^n\left(\frac{z^2}{x}\right)^p$, then $x^{-2}yz^7 = x^{2m-p}.y^{2n-m}.z^{2p-n}$,

$$\therefore \left.\begin{array}{r} 2m-p=-2, \\ 2n-m=1, \\ 2p-n=7, \end{array}\right\} \text{ to find } m, n, \text{ and } p;$$

and from these equations it is easily found that $m=1$, $n=1$, $p=4$. Similarly for y^6z.]

Now the term involving abc^4 in $(a-b+c)^6$ is $\lfloor 6 . \frac{a(-b).c^4}{1.1.\lfloor 4} = -30abc^4$,

$$\text{.......................} \; ab^3c^2 \; \text{.................} \lfloor 6 . \frac{a(-b)^3.c^2}{1.\lfloor 3.\lfloor 2} = -60ab^3c^2.$$

$\therefore$ the terms required are $-\frac{30yz^7}{x^2}$, and $-60y^6z$.

Ex. 50. Find the coefficient of $\frac{1}{x^8}$ in the expansion of $\left(a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\right)^6$.

Since $a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}=a+by+cy^2+dy^3$, if we write y for $\frac{1}{x}$,

$\therefore$ coefficient required = coefficient of y^8 in $(a+by+cy^2+dy^3)^6$.

$$\text{Here} \quad \left.\begin{aligned} p+q+r+s&=6, \\ q+2r+3s&=8, \end{aligned}\right\}$$

p	q	r	s
0	5	0	1
0	4	2	0
1	3	1	1
1	2	3	0
2	2	0	2
2	1	2	1
2	0	4	0
3	0	1	2

$\therefore$ coefficient required

$$=\lfloor 6 . \left\{\frac{b^5d}{\lfloor 5}+\frac{b^4c^2}{\lfloor 4.\lfloor 2}+\frac{ab^3cd}{\lfloor 3}+\frac{ab^2c^3}{\lfloor 2.\lfloor 3}+\frac{a^2b^2d^2}{\lfloor 2.\lfloor 2.\lfloor 2}+\frac{a^2bc^2d}{\lfloor 2.\lfloor 2}+\frac{a^2c^4}{\lfloor 2.\lfloor 4}+\frac{a^3cd^2}{\lfloor 3.\lfloor 2}\right\},$$

$$=6b^5d+15b^4c^2+120ab^3cd+60ab^2c^3+90a^2b^2d^2+180a^2bc^2d+15a^2c^4$$
$$+60a^3cd^2.$$

Ex. 51. Find the coefficient of x^7 in $(a+bx+cx^2+dx^3+\ldots in\ inf.)^4$.

Here $\left.\begin{array}{r} p+q+r+s+t+u+v+w=4, \\ q+2r+3s+4t+5u+6v+7w=7, \end{array}\right\}$ are the equations of conditions.

a	b	c	d	e	f	g	h
p	q	r	s	t	u	v	w
3	0	0	0	0	0	0	1
2	1	0	0	0	0	1	0
2	0	1	0	0	1	0	0
1	2	0	0	0	1	0	0
0	3	0	0	1	0	0	0
2	0	0	1	1	0	0	0
1	1	1	0	1	0	0	0
1	0	2	1	0	0	0	0
1	1	0	2	0	0	0	0
0	1	3	0	0	0	0	0
0	2	1	1	0	0	0	0

$$\therefore \text{ coefficient required} = \underline{|4}.\left\{\frac{a^3h}{\underline{|3}}+\frac{a^2bg}{\underline{|2}}+\frac{a^2cf}{\underline{|2}}+\frac{ab^2f}{\underline{|2}}+\frac{b^3e}{\underline{|3}}+\frac{a^2de}{\underline{|2}}\right.$$

$$\left.+\frac{abce}{1}+\frac{ac^2d}{\underline{|2}}+\frac{abd^2}{\underline{|2}}+\frac{bc^3}{\underline{|3}}+\frac{b^2cd}{\underline{|2}}\right\},$$

$$=\left.\begin{array}{l} 4a^3h+12a^2bg+12a^2cf+12ab^2f+4b^3e+12a^2de \\ +24abce+12ac^2d+12abd^2+4bc^3+12b^2cd \end{array}\right\}.$$

Ex. 52. Find the coefficient of x^3 in $(a+bx+cx^2)^{\frac{3}{2}}$.

Here $\left.\begin{array}{r} p+q+r=\frac{3}{2}, \\ q+2r=3, \end{array}\right\}$ are the equations of condition.

p	q	r
$-\frac{1}{2}$	1	1
$-\frac{3}{2}$	3	0

$\therefore$ coefficient required

$$=\frac{3}{2}\left(\frac{3}{2}-1\right)a^{-\frac{1}{2}}bc+\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\frac{a^{-\frac{3}{2}}b^3}{\underline{|3}},$$

$$=\frac{3}{4}\cdot\frac{bc}{a^{\frac{1}{2}}}-\frac{b^3}{16a^{\frac{3}{2}}}.$$

Ex. 53. Find the coefficient of x^4 in $(1+bx+cx^2+dx^3+\ldots)^{-4}$.

Here $\left.\begin{array}{l} p+q+r+s+t+\ldots=-4, \\ q+2r+3s+4t+\&c.=\ \ 4,\end{array}\right\}$

p	q	r	s	t
-8	4	0	0	0
-7	2	1	0	0
-6	0	2	0	0
-6	1	0	1	0
-5	0	0	0	1

$\therefore$ coefficient required

$$=-4(-5)(-6)(-7)\cdot\frac{b^4}{\underline{|4}}-4(-5)(-6)\cdot\frac{b^2c}{\underline{|2}}-4(-5)\frac{c^2}{\underline{|2}}-4(-5)bd-4e,$$

$$=35b^4-60b^2c+10c^2+20bd-4e.$$

Ex. 54. In the expansion of $\sqrt[3]{a+bx+cx^2+dx^3+ex^4+\ldots}$, find the coefficients of x^3 and x^4.

Here 1st for x^3, $\left.\begin{array}{l} p+q+r+s+\ldots=\frac{1}{3}, \\ q+2r+3s+\ldots=3,\end{array}\right\}.$

a	b	c	d
p	q	r	s
$-\frac{2}{3}$	0	0	1
$-\frac{5}{3}$	1	1	0
$-\frac{8}{3}$	3	0	0

$\therefore$ coefficient required

$$=\frac{1}{3}a^{-\frac{2}{3}}d+\frac{1}{3}\left(\frac{1}{3}-1\right)a^{-\frac{5}{3}}bc+\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\frac{a^{-\frac{8}{3}}b^3}{\underline{\lfloor 3}},$$

$$=\sqrt[3]{a}.\left\{\frac{d}{3a}-\frac{2bc}{9a^2}+\frac{5b^3}{81a^3}\right\}.$$

(2) For x^4, $\left.\begin{array}{l} p+q+r+s+t+\ldots=\frac{1}{3}, \\ q+2r+3s+4t+\ldots=4, \end{array}\right\}.$

a p	b q	c r	d s	e t
$-\frac{2}{3}$	0	0	0	1
$-\frac{5}{3}$	1	0	1	0
$-\frac{5}{3}$	0	2	0	0
$-\frac{8}{3}$	2	1	0	0
$-\frac{11}{3}$	4	0	0	0

$\therefore$ coefficient of $x^4=\frac{1}{3}a^{-\frac{2}{3}}e+\frac{1}{3}\left(\frac{1}{3}-1\right)a^{-\frac{5}{3}}bd+\frac{1}{3}\left(\frac{1}{3}-1\right)\frac{a^{-\frac{5}{3}}c^2}{\underline{\lfloor 2}}$

$$+\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\frac{a^{-\frac{8}{3}}b^2c}{\underline{\lfloor 2}}+\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)\frac{a^{-\frac{11}{3}}b^4}{\underline{\lfloor 4}},$$

$$=\sqrt[3]{a}.\left\{\frac{e}{3a}-\frac{2bd}{9a^2}-\frac{c^2}{9a^2}+\frac{5b^2c}{27a^3}-\frac{10b^4}{243a^4}\right\}.$$

Ex. 55. Find the coefficient of x^n in $(1+x+2x^2+3x^3+\ldots in\ inf.)^2$.

$$(1+x+2x^2+3x^3+\ldots\ldots\ldots\ldots+nx^n+\ldots)\times$$
$$(1+x+2x^2+3x^3+\ldots\ldots\ldots\ldots+nx^n+\ldots)$$
$$\overline{}$$

$$
\begin{aligned}
=1+x+2x^2+3x^3+\dots\dots\dots+nx^n+\dots\\
+x+x^2+2x^3+\dots\dots+(n-1)x^n+\dots\\
+2x^2+2x^3+\dots+2(n-2)x^n+\dots\\
+3x^3+\dots+3(n-3)x^n+\dots\\
\dots\dots\dots\dots\dots\dots\dots\dots\\
+(n-1)x^n+\dots\\
+nx^n+\dots
\end{aligned}
$$

$\therefore$ coefficient required

$$=n+n-1+2(n-2)+3(n-3)+\dots(n-1)\{n-(n-1)\}+n,$$

$$=2n+n\{1+2+3+\dots+(n-1)\}-(1^2+2^2+3^2+\dots+\overline{n-1}|^2),$$

$$=2n+n.\frac{n-1}{2}.n-\left\{\frac{n-1}{6}+\frac{(n-1)^2}{2}+\frac{(n-1)^3}{3}\right\}, \text{ (Art. 295, Ex. 4)}$$

$$=2n+\frac{n^3}{2}-\frac{n^2}{2}-\frac{1}{6}\{n-1+3(n^2-2n+1)+2(n^3-3n^2+3n-1)\},$$

$$=2n+\frac{n^3}{2}-\frac{n^2}{2}-\frac{1}{6}\{2n^3-3n^2+n\},$$

$$=\frac{n^3}{6}+\frac{11n}{6}=\frac{1}{6}(n^3+11n).$$

Ex. 56. In the expansion of $(1+2x+3x^2+\dots)^2$, find the coefficient of x^r.

This may be done by the same method as the last; but more easily as follows:

$$(1+2x+3x^2+\dots)^2=\left\{\frac{1}{(1-x)^2}\right\}^2=(1-x)^{-4},$$

$$=1+4x+4.\frac{4+1}{2}x^2+\dots+\frac{4(4+1)(4+2)+\dots+(4+r-1)}{1\,.\,2\,.\,3\,\dots\dots\dots\,r}.x^r+\&c.$$

$\therefore$ coefficient required

$$=\frac{4.5.6\dots(r+3)}{1.2.3\dots\dots r}=\frac{4.5.6\dots r.(r+1)(r+2)(r+3)}{1.2.3.4.5.6\dots\dots\dots\dots\dots r},$$

$$=\frac{1}{6}(r+1)(r+2)(r+3).$$

Ex. 57. In the expansion of $(a_0+a_1x+a_2x^2+a_3x^3)^4$, find the number of terms.

Since the number of terms in the expansion of any multinomial

$(a_1+a_2+a_3+\ldots+a_r)^n$ is $\dfrac{r(r+1)(r+2)\ldots(r+n-1)}{1\,.\,2\,.\,3\,\ldots\ldots\,n}$, (Art. 324, Cor.)

$$\therefore \text{ number of terms in } (a_0+a_1x+a_2x^2+a_3x^3)^4 = \frac{4.5.6.7}{1.2.3.4},$$
$$=35.$$

Ex. 58. Find the number of terms in the expansion of

$$(a+b+c)^7.$$

Here $r=3$, $n=7$, $\therefore$ substituting in the general formula,

$$\text{number of terms required} = \frac{3.4.5.6.7.8.9}{1.2.3.4.5.6.7} = \frac{8.9}{1.2} = 36.$$

Or it may be done thus,

$$(a+b+c)^7 = \{a+(b+c)\}^7,$$
$$= a^7+7a^6(b+c)+21a^5(b+c)^2+\ldots\ldots+(b+c)^7,$$

from which it is easy to conclude that the

$$\text{number of terms} = 1+2+3+4+\ldots\ldots+8,$$
$$=(1+8)\frac{8}{2}=36.$$

Ex. 10. $1^3=1$, $2^3=3+5$, $3^3=7+9+11$, &c. Write down n^3 after the same law, and verify the result.

In each equality we observe that the number of terms of the right hand member is the number which is cubed—that the first term is that number squared and diminished by the preceding number—and the common difference of the several terms is 2. So that for n^3 the first term will be $n^2-(n-1)$, following the same law; and therefore

$$n^3 = \{n^2-(n-1)\}+\{n^2-(n-3)\}+\{n^2-(n-5)\}+\&\text{c. to } n \text{ terms},$$

$$=\{2(n^2-n+1)+(n-1)2\}\frac{n}{2} = n^3, \text{ as it ought.}$$

EVOLUTION OF SURDS.

Ex. 19. Shew that $\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{\frac{1}{2}}$ is equal to $\frac{\sqrt{2}}{1+\sqrt{3}}$.

$$\frac{2-\sqrt{3}}{2+\sqrt{3}}=\frac{4-2\sqrt{3}}{4+2\sqrt{3}}=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2,$$

$$\therefore \left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{\frac{1}{2}}=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{\frac{1}{2}}=\frac{(\sqrt{3}-1)^{\frac{1}{2}}.(\sqrt{3}+1)^{\frac{1}{2}}}{\sqrt{3}+1}=\frac{\sqrt{2}}{\sqrt{3}+1}.$$

Ex. 24. If $x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}$, and $r=0$, shew that the values of x are 0, and $\pm\sqrt{q}$.

If $r=0$, $x=\sqrt[3]{\sqrt{-\left(\frac{q}{3}\right)^3}}+\sqrt[3]{-\sqrt{-\left(\frac{q}{3}\right)^3}}$,

$$=\sqrt{-\frac{q}{3}}.\sqrt[3]{1}+\sqrt{-\frac{q}{3}}.\sqrt[3]{-1},$$

$$=\sqrt{-\frac{q}{3}}.\left\{\sqrt[3]{1}+\sqrt[3]{-1}\right\}=\sqrt{-\frac{q}{3}}.\{1-1\}=0\ldots\ldots\ \ldots.(1)$$

or $=\sqrt{-\frac{q}{3}}.\left\{\frac{-1\pm\sqrt{-3}}{2}+\frac{1\pm\sqrt{-3}}{2}\right\}$, (*Alg. App.* Ex. 20)

$$=\sqrt{-\frac{q}{3}}.\left(\pm\sqrt{-3}\right)=\pm\sqrt{\frac{q}{3}}.\sqrt{-1}.\sqrt{3}.\sqrt{-1},$$

$$=\mp\sqrt{q}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Ex. 25. If $\left(x-\frac{y}{p}\right).\{b-y-(a-x)p\}=c^2$, and $p=\sqrt{\frac{y^2-by}{x^2-ax}}$, prove that $\sqrt{(b-y)x}-\sqrt{(a-x)y}=c$.

$$\left(x-y\sqrt{\frac{ax-x^2}{by-y^2}}\right).\left\{b-y-\sqrt{\frac{y}{x}(b-y)(a-x)}\right\}=c^2,$$

or $x(b-y)-\sqrt{y}.\sqrt{a-x}.\sqrt{x}.\sqrt{b-y}-\sqrt{x}.\sqrt{a-x}.\sqrt{y}.\sqrt{b-y}+y(a-x)=c^2$,

or $x(b-y)-2\sqrt{xy}.\sqrt{a-x}.\sqrt{b-y}+y(a-x)=c^2$,

or $$\sqrt{(b-y)x}-\sqrt{(a-x)y}=c.$$

Ex. 26. Given $2\{x^2+y^2-x-y\}+1=0$, find the real values of x and y.

$$x^2+y^2-x-y+\frac{1}{2}=0,$$

$$x^2-x+\frac{1}{4}=-\left(y^2-y+\frac{1}{4}\right),$$

$$\therefore\ x-\frac{1}{2}=\left(y-\frac{1}{2}\right)\sqrt{-1},$$

$$\text{or } x-\frac{1}{2}-\left(y-\frac{1}{2}\right)\sqrt{-1}=0,$$

$$\therefore\ x-\frac{1}{2}=0, \text{ and } y-\frac{1}{2}=0,$$

$$\therefore\ x=\frac{1}{2}, \text{ and } y=\frac{1}{2}.$$

Ex. 27. Given $(x+y\sqrt{-1})^2=a+b\sqrt{-1}$, find the real values of x and y.

$$x^2-y^2+2xy\sqrt{-1}=a+b\sqrt{-1},$$

$$\therefore\ \left.\begin{array}{l} x^2-y^2=a, \\ \text{and } 2xy=b, \end{array}\right\}$$

$$\therefore\ x^2-\frac{b^2}{4x^2}=a,$$

$$4x^4-4ax^2=b^2,$$

$$4x^4-4ax^2+a^2=a^2+b^2,$$

$$2x^2=\sqrt{a^2+b^2}+a,$$

$$\therefore\ x=\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)},$$

$$\text{and } y=\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}.$$

Ex. 28. Given $\dfrac{3x+2y\sqrt{-1}}{5\sqrt{-1}-2}=\dfrac{15}{8x+3y\sqrt{-1}}$, find the real values of x and y.

$$24x^2+25xy\sqrt{-1}-6y^2=75\sqrt{-1}-30,$$

$$\therefore\ 24x^2-6y^2=-30, \text{ and } 25xy=75,$$

$$\text{or } 4x^2-y^2=-5, \quad \text{and} \quad xy=3,$$

$$\therefore\ 4x^2-\frac{9}{x^2}=-5,$$

$$4x^4+5x^2=9,$$

$$\therefore\ x=1, \text{ and } y=\frac{3}{x}=3.$$

Ex. 29. Find the relations subsisting between a, b, c, d, when the square root of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ can be expressed in the form $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}$.

Let $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}$,

then $a+\sqrt{b}+\sqrt{c}+\sqrt{d}=\alpha+\beta+\gamma+2\sqrt{\alpha\beta}+2\sqrt{\alpha\gamma}+2\sqrt{\beta\gamma}$,

$$\therefore\ \left.\begin{aligned} \alpha+\beta+\gamma&=a,\\ 2\sqrt{\alpha\beta}&=\sqrt{b},\\ 2\sqrt{\alpha\gamma}&=\sqrt{c},\\ 2\sqrt{\beta\gamma}&=\sqrt{d}, \end{aligned}\right\},$$

and it remains to eliminate the three quantities α, β, γ, betwixt these four equations.

$$1\text{st } \frac{2\sqrt{\alpha\beta}.2\sqrt{\alpha\gamma}}{2\sqrt{\beta\gamma}}=2\alpha=\sqrt{\frac{bc}{d}}.$$

$$\text{Also } \frac{2\sqrt{\alpha\beta}.2\sqrt{\beta\gamma}}{2\sqrt{\alpha\gamma}}=2\beta=\sqrt{\frac{bd}{c}},$$

$$\text{and } \frac{2\sqrt{\alpha\gamma}.2\sqrt{\beta\gamma}}{2\sqrt{\alpha\beta}}=2\gamma=\sqrt{\frac{cd}{b}},$$

$$\therefore\ \sqrt{\frac{bc}{d}}+\sqrt{\frac{bd}{c}}+\sqrt{\frac{cd}{b}}=2(\alpha+\beta+\gamma),$$

$$=2a.$$

Hence the conditions required are that each of the quantities $\sqrt{\frac{bc}{d}}$, $\sqrt{\frac{bd}{c}}$, $\sqrt{\frac{cd}{b}}$, must be rational, and the sum of them equal to $2a$.

INDETERMINATE COEFFICIENTS.

Ex. 16. Resolve $7x^2-6x-1$ into two factors of the first degree.

Since -1 is the last term, which can only arise by the product of $+1$ and -1; assume the two factors to be $Ax+1$, and $Bx-1$.

Then $(Ax+1)(Bx-1)$, or $\left.\begin{array}{l} ABx^2-(A-B)x-1, \\ \text{and } 7x^2-6x-1, \end{array}\right\}$ are identical;

$$\left.\begin{array}{r} \therefore\ AB=7, \\ \text{and } A-B=6, \end{array}\right\}, \text{ to find } A \text{ and } B.$$

$$\left.\begin{array}{l} A^2-2AB+B^2=36, \\ 4AB \quad\quad =28, \end{array}\right\},$$

$$\therefore\ \left.\begin{array}{l} A+B=8, \\ A-B=6, \end{array}\right\},$$

$$\therefore\ A=7, \text{ and } B=1.$$

$\therefore$ the factors required are $7x+1$ and $x-1$.

Ex. 17. Resolve $2x^2-21xy-11y^2-x+34y-3$ into factors of the first degree.

Since the first term $2x^2$ can only be produced by the factors $2x$ and x, assume the two factors to be $2x+Ay+B$, and $x+ay+b$.

Then $2x^2-21xy-11y^2-x+34y-3$ is identical with the product of $(2x+Ay+B)(x+ay+b)$, or $2x^2+(2a+A)xy+Aay^2+(2b+B)x$
$+(Ba+Ab)y+Bb$.

$$\therefore\ \left.\begin{array}{r} 2a+A=-21\ldots(1) \\ Aa=-11\ldots(2) \\ 2b+B=-\ \ 1\ldots(3) \\ Bb=-\ \ 3\ldots(4) \\ Ba+Ab=\ \ 34\ldots(5) \end{array}\right\} \text{ to find } A,\ a,\ B,\ b.$$

From (1) $2a^2 + Aa = -21a$,

by (2) $\therefore\ 2a^2 + 21a = 11$,

$$a^2 + \frac{21}{2}a + \left(\frac{21}{4}\right)^2 = \frac{11}{2} + \frac{441}{16} = \frac{529}{16},$$

$$\therefore\ a = \frac{-21 \pm 23}{4} = -\frac{1}{2}, \text{ or } -11.$$

$\therefore$ for the present purpose $a = -11$, and $\therefore\ A = \dfrac{-11}{a} = 1$.

Similarly, from (3) and (4), $b = 1$, and $B = -3$.

$\therefore$ the factors required are $2x + y - 3$ and $x - 11y + 1$.

CONTINUED FRACTIONS.

Ex. 7. From the last Example (Ex. 6) deduce an explanation of the Julian and Gregorian corrections of the Calendar, having given the true length of the year to be 365·2422638... days.

The excess of the true tropical year over 365 days is 0·2422638... days, which is expressed by the converging fractions

$$\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}, \frac{47}{194}, \text{ \&c.}$$

Now, if the first of these fractions be taken, it signifies that the excess amounts to 1 day in 4 years, which is made up by the common leap year, or *Julian* correction: but this correction is too great.

Again, if the fraction $\frac{47}{194}$ be taken, which is also too great, though much nearer, it signifies that the excess amounts to 47 days in 194 years, or 94 days in 388 years, or 94+3, i.e. 97 days, very nearly, in 388+12, or 400, years. Therefore the *Julian* correction is too great to the amount of 3 days, very nearly, in 400 years—and this is provided for in the *Gregorian* Calendar by the omission of 3 ordinary leap years in 400 years, viz. in those *centurial* years, when the number of *centuries* is not exactly divisible by 4. Thus 1700, 1800, 1900, are not leap years by the Gregorian Calendar; but 2000 is a leap year, because 20 is divisible by 4.

Ex. 13. If $\left(\frac{7}{12}\right)^x = \frac{3}{4}$, find x.

$$\text{Since } \left(\frac{7}{12}\right)^0 = 1 > \frac{3}{4}, \text{ and } \left(\frac{7}{12}\right)^1 = \frac{2\frac{1}{3}}{4} < \frac{3}{4},$$

$\therefore$ x lies between 0 and 1.

Assume $x = \frac{1}{y}$, then $\left(\frac{7}{12}\right)^{\frac{1}{y}} = \frac{3}{4}$, $\therefore$ $\left(\frac{3}{4}\right)^y = \frac{7}{12}$, an equation similar to the proposed one.

$$\text{Again, since } \left(\frac{3}{4}\right)^0 = 1 > \frac{7}{12}; \quad \left(\frac{3}{4}\right)^1 = \frac{9}{12} > \frac{7}{12};$$

$$\left(\frac{3}{4}\right)^2 = \frac{9}{16} < \frac{7}{12}; \quad \therefore y \text{ lies between 1 and 2.}$$

$$\text{Assume } y = 1 + \frac{1}{z}, \text{ then } \left(\frac{3}{4}\right)^{1+\frac{1}{z}} = \frac{7}{12},$$

$$\therefore \left(\frac{3}{4}\right)^{\frac{1}{z}} = \frac{7}{12} \div \frac{3}{4} = \frac{7}{9}, \text{ or } \left(\frac{7}{9}\right)^z = \frac{3}{4}.$$

$$\text{Again, since } \left(\frac{7}{9}\right)^0 = 1 > \frac{3}{4}; \quad \left(\frac{7}{9}\right)^1 > \frac{3}{4}; \quad \left(\frac{7}{9}\right)^2 = \frac{49}{81} < \frac{3}{4},$$

$\therefore$ z lies between 1 and 2.

$$\text{Assume } z = 1 + \frac{1}{u}; \text{ then } \left(\frac{7}{9}\right)^{1+\frac{1}{u}} = \frac{3}{4},$$

$$\therefore \left(\frac{7}{9}\right)^{\frac{1}{u}} = \frac{3}{4} \div \frac{7}{9} = \frac{27}{28}, \quad \therefore \left(\frac{27}{28}\right)^u = \frac{7}{9};$$

and so on, until a sufficient approximation is attained for the purpose required.

$$\text{Then } x = \frac{1}{y} = \frac{1}{1+\frac{1}{z}} = \frac{1}{1+\frac{1}{1+\frac{1}{u}}} = \&c.$$

from which the approximate value of x is easily deduced by the usual method, and is found to be 0·5333... nearly.

Ex. 16. Approximate by continued fractions to the roots of the equations,

$$(1)\ 5x^2-3=0, \qquad (2)\ x^2-5x+3=0.$$

Here (1) $5x^2-3=0$, $\therefore x=\sqrt{\frac{3}{5}}$.

Now $\sqrt{\frac{3}{5}}=\frac{1}{5}\sqrt{15}$,

$$\sqrt{15}=3+\sqrt{15}-3=3+\frac{6}{\sqrt{15}+3}=3+\frac{1}{\frac{\sqrt{15}+3}{6}},$$

$$\frac{\sqrt{15}+3}{6}=1+\frac{\sqrt{15}-3}{6}=1+\frac{1}{\sqrt{15}+3}=1+\frac{1}{3+\sqrt{15}},$$

$$\therefore \sqrt{15}=3+\cfrac{1}{1+\cfrac{1}{3+\sqrt{15}}}=3+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{6+\&c.}}}}$$

$\therefore$ for $\sqrt{15}$ the quotients are 3, 1, 6, 1, 6, &c.

and converging fractions are $\frac{3}{1}$, $\frac{4}{1}$, $\frac{27}{7}$, $\frac{31}{8}$, $\frac{213}{55}$, &c.

$\therefore$ for $\sqrt{\frac{3}{5}}$ the converging fractions are $\frac{3}{5}$, $\frac{4}{5}$, $\frac{27}{35}$, $\frac{31}{40}$, $\frac{213}{275}$, &c.

(2) In $x^2-5x+3=0$, $x=\frac{1}{2}(5\pm\sqrt{13})$.

Now $$\frac{5+\sqrt{13}}{2}=4+\frac{\sqrt{13}-3}{2}=4+\frac{2}{\sqrt{13}+3}=4+\frac{1}{\frac{\sqrt{13}+3}{2}},$$

$$\frac{\sqrt{13}+3}{2}=3+\frac{\sqrt{13}-3}{2}=3+\frac{2}{\sqrt{13}+3}=3+\frac{1}{\frac{\sqrt{13}+3}{2}},$$

$\therefore$ the quotients are 4, 3, 3, 3, &c.

... fractions ... $\frac{4}{1}$, $\frac{13}{3}$, $\frac{43}{10}$, $\frac{142}{33}$, &c.

Also $\dfrac{5-\sqrt{13}}{2}=0+\dfrac{6}{5+\sqrt{13}}=0+\dfrac{1}{\dfrac{5+\sqrt{13}}{6}}$,

$$\frac{5+\sqrt{13}}{6}=1+\frac{\sqrt{13}-1}{6}=1+\frac{2}{\sqrt{13}+1}=1+\frac{1}{\dfrac{\sqrt{13}+1}{2}},$$

$$\frac{\sqrt{13}+1}{2}=2+\frac{\sqrt{13}-3}{2}=2+\frac{2}{\sqrt{13}+3}=2+\frac{1}{\dfrac{\sqrt{13}+3}{2}},$$

$$\frac{\sqrt{13}+3}{2}=3+\frac{\sqrt{13}-3}{2}=3+\frac{2}{\sqrt{13}+3}=3+\frac{1}{\dfrac{\sqrt{13}+3}{2}},$$

∴ the quotients are 0, 1, 2, 3, 3, &c.

... fractions ... $\frac{1}{1}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{23}{33}$, &c.

Ex. 17. Shew that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$, and that it differs from the latter fraction by a quantity less than $\dfrac{1}{2\times 305\times 1292}$.

$$\sqrt{5}=2+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\dots}}}$$ See Art. 347.

∴ Quotients are 2, 4, 4, 4, 4, 4, &c.

and fractions ... $\frac{2}{1}$, $\frac{9}{4}$, $\frac{38}{17}$, $\frac{161}{72}$, $\frac{682}{305}$, $\frac{2889}{1292}$, &c.

Now, by Art. 345, these fractions are alternately less and greater than $\sqrt{5}$, and each of the fractions is nearer to the true value than the next preceding one;

$$\therefore \sqrt{5}>\frac{682}{305}, \text{ and } <\frac{2889}{1292}.$$

Also, since $\sqrt{5}$ lies between these two fractions, and is nearer to the latter than to the former, it differs from the latter fraction by a quantity less than *half* the difference between the two, that is, less than

$$\frac{1}{2}\left(\frac{2889}{1292}-\frac{682}{305}\right), \text{ or } \frac{1}{2\times305\times1292}.$$

INDETERMINATE EQUATIONS AND PROBLEMS.

Ex. 13. Find all the positive integral solutions of

$$xy+x^2=2x+3y+29.$$

Here $(x-3)y=2x+29-x^2$,

$$\therefore y=\frac{2x+20}{x-3}-\frac{x^2-9}{x-3},$$

$$=2+\frac{26}{x-3}-x-3,$$

$$=\frac{26}{x-3}-(x+1).$$

$\therefore$ $x-3$ must divide 26 without remainder, and $\therefore$ can only be 1, or 2, or 13; so that the only possible values of x are 4, 5, and 16. And the corresponding values of y are 21, 7, and -15.

$\therefore$ the values required are $x=4, 5.$ $y=21, 7.$

Ex. 16. Find the number of solutions of $3x+7y+17z=100$ in positive integers.

Here $3x+7y=100-17z$,

and since neither x nor y can be less than 1, $100-17z$ cannot be less than 10;

$\therefore$ $17z$ cannot be greater than 90,

and z .... $\frac{90}{17}$ or $5\frac{5}{17}$,

$\therefore$ all the values of z are found in 1, 2, 3, 4, 5.

Hence the number of solutions required will be the number of solutions of the several equations

$$3x+7y=83\ldots(1)$$

$$3x+7y=66\ldots(2)$$

$$3x+7y=49\ldots(3)$$

$$3x+7y=32\ldots(4)$$

$$3x+7y=15\ldots(5)$$

Now, by Art. 359, it is readily shewn that the number of solutions of (1) is 4; of (2) is 3; of (3) is 2; of (4) is 1; and of (5) is 0:

$$\therefore \text{ Total number required} = 4+3+2+1=10.$$

Ex. 25. Shew that the solution of $ax+by=c$ in positive integers is always possible, if a be prime to b, and $c > ab-(a+b)$.

Let $c = ab-(a+b)+m$, then

$$ax+by=ab-(a+b)+m,$$

$\therefore\ x=b-1-\dfrac{b(y+1)-m}{a}$, which is integral, if $\dfrac{b(y+1)-m}{a}$ be integral. Let $y+1=y'$, and suppose

$$\frac{by'-m}{a}=z,$$

$\therefore\ az-by'=-m$, which always admits of positive integral solutions (Art. 359, Cor. 3), that is, y' and z, or y and z are positive integers.

$\therefore$ Also $x=b-1-z$, is integral.

But the general value of z in $az-by'=-m$ is $\mp mq+bt$, (Art. 358, Cor. 1), where *t may have any integral value positive or negative.* Taking the upper sign, $z=-mq+bt$, and if $t=1$, $z=b-mq$, which $<b$, if m be *positive*. Taking the lower sign, $z=mq+bt$; and for negative values of t, $z=mq-bt$, which $<b$ for all values of t greater than $\dfrac{mq}{b}-1$. Therefore in either case there will be one or more values of z less than b; and $\therefore\ x=b-1-z$, is *positive* as well as integral. Hence x and y are both positive integers, i.e. the equation $ax+by=c$, has positive integral solutions, if m be positive, that is, if $c > ab-(a+b)$.

Ex. 27. A certain sum consists of x£ y shillings; and its half of y£ x shillings; find the sum.

The whole sum, in shillings, $=20x+y$,

$$\therefore \frac{1}{2}(20x+y)=20y+x,$$

$$18x=39y,$$

$$\therefore x=2y+\frac{y}{6}, \text{ a whole number.}$$

$\therefore$ y must be either 6, or a multiple of 6: but, as x is in the shillings' place in one of the given expressions, $x<20$, $\therefore$ $y=6$. And then $x=13$.

Hence the sum required is £13 6s.

Ex. 29. What value of x will make ax^2+bx+c^2 a complete square?

$$\text{Assume } ax^2+bx+c^2=\left(\frac{m}{n}x+c\right)^2,$$

$$=\frac{m^2}{n^2}x^2+\frac{2m}{n}cx+c^2,$$

$$\left(\frac{m^2}{n^2}-a\right)x^2=\left(b-\frac{2cm}{n}\right)x,$$

$$\therefore x=\frac{bn^2-2cmn}{m^2-an^2}.$$

Ex. 30. What integral values of x will make $2x^2+x+8$ a complete square?

Assume $x=p-1$, then

$$2x^2+x+8=2(p^2-2p+1)+p-1+8=\text{a square},$$

$$2p^2-3p+9=\text{a square},$$

$$=(pq-3)^2, \text{ suppose},$$

$$=p^2q^2-6pq+9,$$

$$\therefore p(q^2-2)=3(2q-1),$$

$$\therefore p=\frac{3(2q-1)}{q^2-2}.$$

If $q=-1$, then $p=9$, $x=8$, and the square is $(12)^2$;

... $q=1$, then $p=-3$, $x=-4$, ... 6^2;

... $2q=1$, then $p=0$, $x=-1$, ... 3^2;

... $2q=3$, then $p=24$, $x=23$, ...$(33)^2$.

Ex. 31. What value of b will make b^2-4ac a complete square?

$$\text{Assume } b^2-4ac=(b-2am)^2,$$
$$=b^2-4abm+4a^2m^2,$$
$$4am.b=4a^2m^2+4ac,$$
$$\therefore\ b=am+\frac{c}{m}.$$

Ex. 32. Find three *square* numbers in Arithmetic Progression.

Let x^2, y^2, z^2, be the numbers, then

$$x^2-y^2=y^2-z^2,$$
$$\text{or } (x+y)(x-y)=(y+z)(y-z).$$
$$\left.\begin{aligned}\text{Assume } m(x+y)&=n(y+z)\ldots(1)\\ \text{and } n(x-y)&=m(y-z)\ldots(2)\end{aligned}\right\}$$

then from (1), $(m-n)y=nz-mx$,

$$\therefore\ y=\frac{nz-mx}{m-n}.$$

Again, from (2), $(m+n)y=nx+mz$,

$$\therefore\ y=\frac{nx+mz}{m+n}.$$

$$\therefore\ \frac{nx+mz}{m+n}=\frac{nz-mx}{m-n},$$

$$mnx+m^2z-n^2x-mnz=mnz-m^2x+n^2z-mnx,$$

$$\therefore\ (m^2-n^2+2mn)x=-(m^2-n^2-2mn)z.$$

If then $z=m^2-n^2+2mn$, $x=-(m^2-n^2-2mn)$,

and $y^2=\dfrac{x^2+z^2}{2}=(m^2-n^2)^2+4m^2n^2=(m^2+n^2)^2$.

$\therefore$ the three squares required are

$$(m^2-n^2-2mn)^2,\quad (m^2+n^2)^2,\quad (m^2-n^2+2mn)^2.$$

SCALES OF NOTATION.

Ex. 12. In what scale of notation will a number that is double of 145 be expressed by the same digits?

Let r be the radix of the scale required; then

$$1\times r^2+4r+5=290,$$

$$r^2+4r+4=289,$$

$$r+2=17,$$

$$\therefore\ r=15.$$

Ex. 25. A certain number consists of two digits, such that when the digits are reversed the number is divisible by 3, and is to the former number as 23 : 32. Required the number.

Let x and y be the digits; then

by the question, $10x+y : 10y+x :: 32 : 23,$

$$230x+23y=320y+32x,$$

$$198x=297y,$$

$$2x=3y.$$

Also $\frac{10y+x}{3}$ or $\frac{20y+2x}{6}$ is a whole number,

or $\frac{23y}{6}$ ....

But y cannot be greater than 9, $\therefore\ y=6$;

and $x=\frac{3y}{2}=9$,

$\therefore$ number required is 96.

Ex. 27. If N, N', be any two numbers in the denary scale composed of the same digits differently arranged, prove that $N\sim N'$ is divisible by 9.

Let a_0, a_1, a_2, a_3, &c. be the digits; so that

$N=a_0+10a_1+10^2a_2+10^3a_3+$&c. of which the general term is $10^m a_m$; then with the same digits differently arranged to make

N', the general term containing a_m will be $10^n a_m$; and $\therefore$ the general term of $N \sim N'$ will be

$$(10^m \sim 10^n)a_m = 10^n(10^{m-n}-1)a_m, \text{ if } m > n,$$

$$\text{or } = 10^m(10^{n-m}-1)a_m, \text{ if } m < n.$$

But since $10^p - 1 = 999$ &c. to p digits, whatever be the value of p, and is therefore divisible by 9,

$\therefore$ $N \sim N'$ is divisible by 9.

PROPERTIES OF NUMBERS.

Ex. 1. Prove that n^3 divided by 4 cannot leave 2 for a remainder, n being any of the natural numbers.

Every number is of one of the forms $4m$, $4m+1$, $4m+2$, $4m+3$.

$$\text{Now } (4m)^3 = 64m^3 \ldots\ldots\ldots\ldots(1)$$

$$(4m+1)^3 = 64m^3 + 48m^2 + 12m + 1 \ldots\ldots(2)$$

$$(4m+2)^3 = 64m^3 + 96m^2 + 48m + 8 \ldots\ldots(3)$$

$$(4m+3)^3 = 64m^3 + 144m^2 + 108m + 27 \ldots\ldots(4)$$

and (1) divided by 4 leaves a remainder 0,

(2) 1,

(3) 0,

(4) .... 3.

$\therefore$ *no* cube number divided by 4 leaves a remainder 2.

Ex. 6. If each of the quantities a, b, n, be a whole number, shew that $\{2a+(n-1)b\}\frac{n}{2}$ is a whole number.

1st. If n be even, then $\frac{n}{2}$ is a whole number, and $\therefore$ also the proposed quantity.

2nd. If n be odd, then $n-1$ is even, and $\therefore$ $2a+(n-1)b$ is divisible by 2, and $\therefore$ the proposed quantity is a whole number.

Ex. 8. Shew that x^5-5x^3+4x is divisible by 120, whatever positive whole number x may be.

Here $x^5-5x^3+4x=x^3(x^2-4)-x(x^2-4)$,

$$=(x^3-x)(x^2-4),$$
$$=x(x^2-1)(x^2-4),$$
$$=x(x-1)(x+1)(x-2)(x+2),$$
$$=(x-2)(x-1)x(x+1)(x+2)=\text{the product of}$$

5 consecutive numbers, and is $\therefore$ divisible by 1.2.3.4.5 or 120.

Ex. 9. Shew that $\frac{x+1}{12}(2x^2+x+3)$ is a whole number, if x be odd.

Every odd number may be represented by $2n+1$; substituting this for x, the proposed quantity becomes

$$\frac{2n+2}{12}\{2(2n+1)^2+2n+1+3\},$$
$$=\frac{n+1}{6}(8n^2+8n+2+2n+4),$$
$$=\frac{n+1}{6}(8n^2+10n+6),$$
$$=(n+1)\left(n^2+n+1+\frac{n^2+2n}{3}\right),$$
$$=(n+1)(n^2+n+1)+\frac{(n+1)n(n+2)}{3}.$$

Now $n(n+1)(n+2)$, being the product of 3 consecutive numbers, is divisible by 1.2.3, and $\therefore$ by 3. Hence the proposed quantity is a whole number.

Ex. 14. If n be a prime number greater than 3, $\frac{n^2-1}{24}$ is an integer.

Every prime number greater than 3 is of one of the forms $6m+1$, $6m-1$ (Art. 374); $\therefore$ $n=6m\pm1$, and

$$(6m\pm1)^2-1=36m^2\pm12m=12m(3m\pm1),$$
$$=12m(m\pm1)+24m^2;$$

but $m(m\pm1)$ is divisible by 2, (Art. 369),

$\therefore$ $(6m\pm1)^2-1$ 24,

or $\frac{n^2-1}{24}$ is an integer.

Ex. 16. If there be two binomials each of which is the sum of two squares, their product is the sum of two squares.

Let a^2+b^2, c^2+d^2 be the two binomials; then

$$a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1}),$$

$$c^2+d^2=(c+d\sqrt{-1})(c-d\sqrt{-1}),$$

$$\therefore\ (a^2+b^2)(c^2+d^2)=(a+b\sqrt{-1})(c+d\sqrt{-1})(a-b\sqrt{-1})(c-d\sqrt{-1}),$$

$$=\{(ac-bd)+(ad+bc)\sqrt{-1}\}\{(ac-bd)-(ad+bc)\sqrt{-1}\},$$

$$=(ac-bd)^2+(ad+bc)^2.$$

Ex. 17. Neither the sum nor the difference of two irreducible fractions, whose denominators are different, can be an integer.

Let $\frac{a}{b}$, $\frac{c}{d}$ be the fractions, a prime to b, and c to d;

then $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$.

Suppose t to be the Greatest Common Measure of b and d; and let $b=tB$, $d=tD$; then B and D are prime to each other; and

$$\frac{ad\pm bc}{bd}=\frac{aD\pm cB}{tBD}.$$

Now, if this fraction can be equal to a whole number, the numerator must be divisible by the denominator, and $\therefore$ by each of its prime factors. But, taking the factor D, aD is divisible by it, and $\therefore$, if the fraction is equal to a whole number cB must of itself be divisible by D: and B and D are prime to each other, and so also are c and d, and $\therefore$ c and D; $\therefore$ cB is *not* divisible by D, and the fraction is *not* equal to a whole number.

Ex. 18. If n be any number, and a the difference between n and the next greater square number, and b the difference between n and the next less square number, shew that $n-ab$ is a square.

Let m^2 be the square number less than n,

$(m+1)^2$ .... greater ...

$$\therefore\ a=(m+1)^2-n,$$

$$b=n-m^2,$$

$$\therefore\ ab=(m+1)^2n-n^2-m^2(m+1)^2+m^2n,$$
$$=m^2n+2mn+n-n^2-m^4-2m^3-m^2+m^2n,$$
$$\therefore\ n-ab=m^4+m^2+n^2+2m^3-2m^2n-2mn,$$
$$=(m^2+m-n)^2,$$
$$=\text{a square number, i.e. the square of } m^2+m-n.$$

Ex. 19. The product of two different primes cannot be a square.

Let p and q represent any two *different* prime numbers; and, if possible, let $pq=m^2$, or $\frac{pq}{m}=m$. Now a number can only be divisible by those prime factors by the multiplication of which it is produced; $\therefore$ p and q being the only prime factors of pq, and pq being divisible by m, either $m=p$, or $m=q$. Suppose $m=p$, then $\frac{pq}{m}=\frac{pq}{p}=q$; but $\frac{pq}{m}=m$; $\therefore m=q$. But $m=p$, $\therefore p=q$, which is impossible, $\because$ p and q are *different* numbers, by the original supposition. Hence $pq=m^2$ is impossible; that is, the product of two *different* primes cannot be a square.

Ex. 23. If N is a number of the form a^mb^n, where a and b are prime numbers, shew that $N.\frac{a-1}{a}.\frac{b-1}{b}$ is the number of integers not greater than N and prime to it.

No number can be represented by a^mb^n which is not found in the series $1, 2, 3, 4, 5, \ldots\ldots a^mb^n$, the number of terms being a^mb^n, or N.

Also a being a prime, there is no number in the series divisible by a, except the a^{th}, $2a^{\text{th}}$, $3a^{\text{th}}, \ldots (a^{m-1}b^n)a^{\text{th}}$ terms, the number of such terms being $a^{m-1}b^n$, or $\frac{N}{a}$.

Again, b being a prime, there is no number in the series divisible by b, except the b^{th}, $2b^{\text{th}}$, $3b^{\text{th}}$, $\ldots (a^mb^{n-1})b^{\text{th}}$ terms, the number of such terms being a^mb^{n-1}, or $\frac{N}{b}$.

And similarly the terms divisible by ab are the $(ab)^{\text{th}}$, $2(ab)^{\text{th}}$, $3(ab)^{\text{th}}, \ldots (a^{m-1}b^{n-1})(ab)^{\text{th}}$, the number of the terms being $a^{m-1}b^{n-1}$, or $\frac{N}{ab}$.

But since every number divisible by ab is divisible both by a and b, $\therefore$ the number divisible by ab are included in each of the former numbers, $a^{m-1}b^n$ divisible by a, and a^mb^{n-1} divisible by b.

Hence n°. of numbers divisible by a only is $\frac{N}{a}-\frac{N}{ab}$,

.... b ...... $\frac{N}{b}-\frac{N}{ab}$,

.... ab ...... $\frac{N}{ab}$.

And the whole n°. of numbers is $a^m b^n$, or N,

$\therefore$ the n°. not greater than N and prime to it $= a^m b^n -$ the n°. of numbers which have any common divisor with $a^m b^n$,

$$= N - \left(\frac{N}{a}-\frac{N}{ab}\right) - \left(\frac{N}{b}-\frac{N}{ab}\right) - \frac{N}{ab},$$

$$= \frac{N}{ab}\{ab-(b-1)-(a-1)-1\},$$

$$= \frac{N}{ab}\{ab-(a+b)+1\},$$

$$= \frac{N}{ab}\{\overline{a-1}\,.\,\overline{b-1}\},$$

$$= N.\frac{a-1}{a}.\frac{b-1}{b}.$$

Ex. 24. If one number (A) have exactly as many places of figures as another (B), and also have more than the first half of its figures identical with the corresponding figures in B, shew that the difference between $\sqrt[n]{A}$ and $\sqrt[n]{B}$ will be less than $\frac{1}{n}$, if n be any whole number not less than 2.

$$\left.\begin{array}{l}\text{Let } A=c+a,\\ \quad\;\; B=c+b,\end{array}\right\}\ a>b, \text{ and each contain } m \text{ digits;}$$

then $$A^{\frac{1}{n}}-B^{\frac{1}{n}} = \frac{A-B}{A^{\frac{n-1}{n}}+A^{\frac{n-2}{n}}B^{\frac{1}{n}}+A^{\frac{n-3}{n}}B^{\frac{2}{n}}+\&\text{c. to } n \text{ terms}},$$

$< \dfrac{A-B}{nB^{\frac{n-1}{n}}}$, A being greater than B, and n a whole number not less than 2,

$$< \frac{1}{n}.\frac{(A-B)B^{\frac{1}{n}}}{B}.$$

Now B contains more than $2m$ figures, suppose $2m+p$, therefore the number of figures in $B^{\frac{1}{n}} = \frac{2m+p}{n}$, if that be integral, or the whole number next greater than $\frac{2m+p}{n}$, if it is not; suppose

$\frac{2m+p+p'}{n}$, where $p'<n$. And the number of figures in $A-B=$ the number in $a-b$, that is, is not greater than m, $\therefore$ the number of figures in $(A-B)B^{\frac{1}{n}}$ is not greater than $\frac{2m+p+p'}{n}+m$. (Art. 371). But n is not less than 2, and the greater n is, the less is $\frac{2m+p+p'}{n}+m$; suppose $n=2$, then the number of figures in $(A-B)B^{\frac{1}{n}}$ is not greater than $2m+p-\frac{p-p'}{2}$; and the number in B is $2m+p$; $\therefore$ since $\frac{p-p'}{2}$ is positive, for p is not less than 1, and $p'<2$, when $n=2$, and *à fortiori* when $n>2$, $(A-B)B^{\frac{1}{n}}$ has fewer figures than B, and $\frac{(A-B)B^{\frac{1}{n}}}{B}<1$, and $\therefore \sqrt[n]{A}-\sqrt[n]{B}<\frac{1}{n}$.

CONVERGING AND DIVERGING SERIES.

Ex. 2. Shew that the series for $(1+x)^n$ given by the Binomial Theorem will always be "*convergent*" when $x<1$; and determine after how many terms the convergency will begin.

Here the ratios of the several terms to those immediately preceding are

$$nx,\quad \frac{n-1}{2}x,\quad \frac{n-2}{3}x,\quad \frac{n-3}{4}x,\ \&c.$$

the general form being $\frac{\text{the }(r+2)^{\text{th}}\text{ term}}{\text{the }(r+1)^{\text{th}}\text{ term}}=\frac{n-r}{r+1}x$.

Now when $r>n$, this ratio is always negative, which shews that the quantities compared have different signs; that is, *the terms of the series are alternately positive and negative* from that point.

Also the *value* of the general ratio, independently of its sign, will be

$$\frac{r-n}{r+1}x,\quad \text{or}\ \frac{rx}{r+1}-\frac{nx}{r+1},\quad \text{or}\ \frac{x}{1+\frac{1}{r}}-\frac{nx}{r+1},$$

of which, as r increases, the second term diminishes continually without limit, and the first has a limit x, which <1. Therefore from a certain point the ratio of any term to the preceding term

LOGARITHMS.

Ex. 15. Given $x^y = y^x$, and $x^p = y^q$, find x and y.

Here $y.\log x = x.\log y$, and $p.\log x = q.\log y$,

$$\therefore \frac{y}{x} = \frac{p}{q},$$

$$\text{and } \therefore \log y - \log x = \log\frac{p}{q},$$

$$\text{or } \frac{p}{q}.\log x - \log x = \log\frac{p}{q},$$

$$\frac{p-q}{q}.\log x = \log\frac{p}{q},$$

$$\log x = \frac{q}{p-q}.\log\frac{p}{q} = \log\left(\frac{p}{q}\right)^{\frac{q}{p-q}},$$

$$\therefore x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}.$$

$$\text{And } y = \frac{p}{q}x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}+1} = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}.$$

Or, without the use of logarithms, as follows:

$$\text{Since } x^y = y^x, \quad \therefore x^{\frac{y}{x}} = y = x^{\frac{p}{q}},$$

$$\therefore \frac{y}{x} = \frac{p}{q}.$$

$$\text{or } \frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}.$$

$$\therefore x^{\frac{p}{q}-1} = x^{\frac{p-q}{q}} = \frac{p}{q},$$

$$\therefore x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}.$$

$$\text{And } y = \frac{p}{q}x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}+1} = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}.$$

Ex. 16. Given $3^{2x}.5^{3x-4}=7^{x-1}.11^{2-x}$, find x.

$$2x.\log 3+(3x-4).\log 5=(x-1).\log 7+(2-x).\log 11,$$

$$(2\log 3+3\log 5+\log 11-\log 7)x=2\log 11+4\log 5-\log 7,$$

$$\left.\begin{Bmatrix}\left.\begin{matrix}0\cdot95424\\+2\cdot09691\\+1\cdot04139\end{matrix}\right]-0\cdot84510\end{Bmatrix}\right..x=\left.\begin{matrix}2\cdot08278\\+2\cdot79588\end{matrix}\right\}-0\cdot84510,$$

$$(4\cdot09254-0\cdot84510)x=4\cdot87866-0\cdot84510,$$

$$\therefore\ x=\frac{4\cdot03356}{3\cdot24744}=1\cdot242073\ldots$$

Ex. 17. Given $(a^4-2a^2b^2+b^4)^{x-1}=\dfrac{(a-b)^{2x}}{(a+b)^2}$, find x.

$$\text{Here}\ (a^2-b^2)^{2x-2}=\frac{(a-b)^{2x}}{(a+b)^2},$$

$$\frac{(a^2-b^2)^{2x}}{(a^2-b^2)^2}=\frac{(a-b)^{2x}}{(a+b)^2},$$

$$(a+b)^{2x}=\left(\frac{a^2-b^2}{a+b}\right)^2=(a-b)^2,$$

$$(a+b)^x=a-b,$$

$$\therefore\ x.\log(a+b)=\log(a-b),$$

$$\therefore\ x=\frac{\log(a-b)}{\log(a+b)}.$$

Ex. 18. Given $5^{x+1}-5^{x-3}+14\frac{1}{9}=4739-5^{x-4}+\dfrac{2}{7}-5^{x-2}$, find x.

$$5^{x+1}+5^{x-2}-5^{x-3}+5^{x-4}=4739+\frac{2}{7}-14\tfrac{1}{9},$$

$$5^x.\left\{5+\frac{1}{5^2}+\frac{1}{5^4}-\frac{1}{5^3}\right\}=4725\tfrac{11}{63},$$

$$5^x.\left\{5+\frac{4}{10^2}+\frac{16}{10^4}-\frac{8}{10^3}\right\}=4725\tfrac{11}{63},$$

$$5^x.\{5+0\cdot04+0\cdot0016-0\cdot008\}=4725\cdot17,$$

$$5^x=\frac{4725\cdot17}{5\cdot0336}=938\cdot72,$$

$$\therefore\ x=\frac{\log 938{\cdot}72}{\log 5}=\frac{2{\cdot}97254}{0{\cdot}69897}=4{\cdot}25.$$

Ex. 19. Given $\left.\begin{array}{r}5\times 3^x-3\times 2^y=30000,\\ \text{and } 3\times 3^x+6\times 2^y=20000,\end{array}\right\}$ find x and y.

From (2) $\qquad 10\times 3^x-6\times 2^y=60000,$

$$\therefore\ 13\times 3^x=80000,$$

$$3^x=\frac{80000}{13},$$

$$x.\log 3=\log 80000-\log 13,$$

$$=4{\cdot}90309-1{\cdot}11394,$$

$$\therefore\ x=\frac{3{\cdot}78915}{0{\cdot}47712}=7{\cdot}94.$$

Again, $\quad 3\times 2^y=5\times 3^x-30000=\dfrac{400000}{13}-30000=\dfrac{10000}{13},$

$$\therefore\ y.\log 2=\log 10^4-\log 39=4-1{\cdot}59106=2{\cdot}40894,$$

$$\therefore\ y=\frac{2{\cdot}40894}{0{\cdot}30103}=8{\cdot}002.$$

Ex. 20. Given $3^{x^2-4x+5}=1200$, find x.

Dividing by 3, $\qquad 3^{x^2-4x+4}=400,$

$$(x^2-4x+4)\log 3=\log 400=\log 100+\log 4,$$

$$=2+0{\cdot}60206,$$

$$\therefore\ x^2-4x+4=\frac{2{\cdot}60206}{0{\cdot}47712}=5{\cdot}45,$$

$$x-2=\pm 2{\cdot}33,$$

$$\therefore\ x=4{\cdot}33,\quad \text{or } -0{\cdot}33.$$

Ex. 21. Given $a^1.a^3.a^5.a^7.\&c.=p$, find the number of factors a^1, a^3, a^5, &c.

Let x = the number required; then

$$a^1.a.^3a^5\ldots a^{2x-1}=p,$$

$$a^{1+3+5\ldots\ldots+2x-1}=p,$$

$$\therefore \{1+3+5+\ldots+(2x-1)\}\log a=\log p,$$

$$(1+2x-1)\frac{x}{2}=\frac{\log p}{\log a}, \text{ or } x^2=\frac{\log p}{\log a},$$

$$\therefore x=\sqrt{\frac{\log p}{\log a}}.$$

INTEREST AND ANNUITIES.

Ex. 5. Prove that the amount of 1£ in n years at compound interest is given within less than a farthing by the first four terms of the expansion of $(1+r)^n$£, the rate of interest being not greater than 4 per cent., and n not greater than 10.

The amount of 1£ at compound interest for n years, (Art. 403) is $(1+r)^n$£, r being the interest of 1£ for one year,

$$=1+nr+\frac{n(n-1)}{2}r^2+\&c\ldots\ldots+r^n,$$

and the sum of the terms in this expansion, after the first four, will obviously be greatest, within the limits permitted, when $n=10$, and $r=0{\cdot}04$. Therefore this will be the case most unfavourable to the truth of the proposition, so that if this be proved true, *à fortiori* it is true in other cases.

$$\text{Now } (1+0{\cdot}04)^{10}=1+10\times0{\cdot}04+\frac{10\times9}{1\times2}(0{\cdot}04)^2+\frac{10\times9\times8}{1\times2\times3}(0{\cdot}04)^3+\&c.$$

$$\text{the } 5^{\text{th}} \text{ term}=\frac{10\times9\times8\times7}{1\times2\times3\times4}\cdot\frac{4^4}{10^8}\cdot960 \text{ farthings},$$

$$=\frac{210\times4^4\times960}{10^8}=\frac{516096}{10^6}=0{\cdot}516096,$$

$$\text{the } 6^{\text{th}} \text{ term}=\frac{6}{5}\times\frac{4}{10^2}\cdot0{\cdot}516096=0{\cdot}024773,$$

$$\text{the } 7^{\text{th}} \text{ term}=\frac{4^2}{10^4}\times0{\cdot}516096=0{\cdot}000826,$$

$$\text{the } 8^{\text{th}} \text{ term}=\frac{4}{7}\times\frac{4}{10^2}\times0{\cdot}000826=0{\cdot}000018,$$

and the 9^{th}, 10^{th}, and last terms being still less than the 8^{th}, it is plain that the sum of all the terms after the 4^{th} will not amount

to 1 farthing. Hence the first four terms of the expansion will give the amount required within less than a farthing.

Ex. 15. An annuity of 20£ for 21 years is sold for 220£; required the rate of interest allowed to the purchaser.

(This is much too difficult for the position which it here occupies, but it is solved as follows:—)

Let x be the *amount* of 1£ for one year; then, since the purchase money, at compound interest for 21 years, should become what would be due for the annuity, if left unpaid for that time,

$$220x^{21}=20(1+x+x^2+\ldots+x^{20}),$$

$$\text{or } 11x^{21}=\frac{x^{21}-1}{x-1}, \text{ or } 11x^{21}-\frac{x^{21}-1}{x-1}=0;$$

from which equation x is to be found.

By the tentative method, and the aid of the Logarithmic Tables, x is shewn to lie between 1·0681 and 1·0682; and 1·0682 being taken for the value of x, the year's interest on £1 is ·0682£, and the required rate per cent. is 6·82, or £6. 16*s.* 5*d.* nearly.

Ex. 17. An annuity A is to commence at the end of p years, and to continue q years; find the equivalent annuity to commence immediately and to continue q years.

Let x be the annuity required.

The present value of $A=\frac{AR^{-p}}{R-1}.\{1-R^{-q}\}$, Art. 421.

.... $x=\frac{x}{R-1}.\{1-R^{-q}\}$, making $p=0$,

$\therefore$ by the question, $\frac{x}{R-1}\{1-R^{-q}\}=\frac{AR^{-p}}{R-1}\{1-R^{-q}\}$,

$$\text{or } x=AR^{-p}=\frac{A}{(1+r)^p}.$$

Ex. 19. A person puts out P£ at interest, and adds to his capital at the end of every year $\frac{1}{m}$th part of the interest for that year; find the amount at the end of n years.

Let r be the interest of 1£ for 1 year; then the addition from another source of $\frac{1}{m}$ of this interest causes each pound to become $1+r+\frac{r}{m}$, and therefore the whole number of pounds to be multiplied

by $1+r+\dfrac{r}{m}$ *yearly*, or by $\left(1+r+\dfrac{r}{m}\right)^n$ in n years. Hence, the amount of P, at the end of n years, will be

$$P\left(1+r+\frac{r}{m}\right)^n,$$

$$\text{or } P\left(\frac{m+mr+r}{m}\right)^n.$$

Ex. 21. A person spends in the first year m times the interest of his property; in the second $2m$ times that of the remainder; in the third $3m$ times that of what remained at the end of the second; and so on. At the end of $2p$ years he has nothing left. Shew that in the p^{th} year he spends as much as he has left at the end of that year.

Let P_n = his principal at the end of n years; then in the next year his interest $=P_n r$, and his expenditure $=(n+1)m\,.\,P_n r$,

$$\therefore \left.\begin{array}{c}\text{his Principal at the end}\\ \text{of the } (n+1)^{\text{th}} \text{ year}\end{array}\right\} = P_n + P_n r - (n+1)m\,.\,P_n r,$$

$$= P_n\,.\,\{1+r-(n+1)mr\}.$$

Now, putting $2p$ for $n+1$, by the question,

$$P_{2p-1}\,.\,\{1+r-2pmr\}=0,$$

$$\therefore\ 1+r=2pmr.$$

But his expenditure in the p^{th} year $=pm\,.\,P_{p-1}r$,

and ... Principal at the end of...... $=P_{p-1}\,.\,\{1+r-pmr\}$

$$=pm\,.\,P_{p-1}r,\quad \because\ 1+r=2pmr,$$

$$=\text{expenditure in } p^{\text{th}} \text{ year}.$$

Ex. 23. A person puts his whole fortune P£ out at interest, at the rate of r£ per 1£ per annum, and requires for his annual expences p£ more than the whole interest of P£. In how many years at this rate will he have spent the whole? and if when he has spent half his capital, he diminishes his expenditure one half, how much longer on this than on the former supposition will he continue solvent?

1st. Let P_n = his Principal at the end of the n^{th} year; then

$$P_n = \text{amount of } P_{n-1} \text{ for 1 year} - (Pr+p),$$

$$=P_{n-1}R-A, \text{ if } R \text{ be put for } 1+r, \text{ and } A \text{ for } Pr+p.$$

Similarly, $P_{n-1}=P_{n-2}R-A,$

$$P_{n-2}=P_{n-3}R-A,$$

$$\ldots=\ldots\ldots\ldots$$

$$P_2=P_1R-A,$$

$$P_1=PR-A.$$

$$\therefore P_n=P_{n-2}R^2-AR-A,$$

$$=P_{n-3}R^3-AR^2-AR-A,$$

$$=P_{n-4}R^4-AR^3-AR^2-AR-A$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=PR^n-A(R^{n-1}-R^{n-2}-\&c.\ldots-R-1),$$

$$=PR^n-A.\frac{R^n-1}{R-1},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

$$\therefore \text{ when } P_n=0,\ PR^n=A.\frac{R^n-1}{R-1},$$

$$\therefore PrR^n=PrR^n+pR^n-(Pr+p),$$

$$\therefore pR^n=Pr+p,$$

$$\therefore \log p+n\log(1+r)=\log(Pr+p),$$

$$\therefore n=\frac{\log(Pr+p)-\log p}{\log(1+r)} \ldots\ldots\ldots\ldots\ldots \quad (2).$$

2ndly. The time of spending the first half of P, by putting $\frac{P}{2}$ for P_n in (1), is easily found; and the difference between this and (2) gives the time of spending the latter half of P, *on the first supposition*, and is

$$\frac{\log(Pr+p)-\log\left(\frac{Pr}{2}+p\right)}{\log(1+r)}.$$

And the time of spending $\frac{P}{2}$, *on the second supposition*, is, from (2),

$$\frac{\log\frac{Pr+p}{2}-\log\frac{p}{2}}{\log(1-r)} \text{ or } \frac{\log(Pr+p)-\log p}{\log(1+r)};$$

$$\therefore \text{the difference required} = \frac{\log\left(\frac{Pr}{2}+p\right)-\log p}{\log(1+r)}.$$

Ex. 29. If two joint proprietors have an equal interest in a freehold estate worth p£ per annum, but one of them purchase the whole to himself by allowing the other an equivalent annuity of q£ for n years; find the relation of p to q.

$$\left.\begin{array}{r}\text{Present value of a freehold}\\ \text{producing } \frac{p}{2}\text{£ per annum}\end{array}\right\} = \frac{p}{2} \div r = \frac{p}{2r}, \quad \text{Art. 420.}$$

$$\left.\begin{array}{r}\text{Present value of an Annuity of}\\ q\text{£ to continue for } n \text{ years}\end{array}\right\} = q \cdot \frac{1-(1+r)^{-n}}{r}, \quad \text{Art. 418.}$$

$$\therefore \frac{p}{2r} = q \cdot \frac{1-(1+r)^{-n}}{r},$$

$$\text{or } \frac{p}{q} = 2\left\{1 - \frac{1}{(1+r)^n}\right\}.$$

Ex. 30. If P represents the population of any place at a certain time, and every year the number of deaths is $\frac{1}{p}$th, and the number of births $\frac{1}{q}$th, of the whole population at the beginning of that year: required the amount of population at the end of n years from that time.

At the end of 1 year from the time the population was P,

$$\text{the increase} = \frac{P}{q} - \frac{P}{p} = P \cdot \frac{p-q}{pq}.$$

$$\therefore P_1, \text{ or population at the end of 1st year} = P + P \cdot \frac{p-q}{pq} = P\left\{1 + \frac{p-q}{pq}\right\}.$$

Similarly,

$$P_2 \ldots\ldots\ldots\ldots \text{2nd} \ldots = P_1\left\{1 + \frac{p-q}{pq}\right\} = P\left\{1 + \frac{p-q}{pq}\right\}^2,$$

$$P_3 \ldots\ldots\ldots\ldots \text{3rd} \ldots = P_2\left\{1 + \frac{p-q}{pq}\right\} = P\left\{1 + \frac{p-q}{pq}\right\}^3;$$

and so on. Therefore

$$P_n \ldots\ldots\ldots\ldots n^{\text{th}} \ldots = P\left\{1 + \frac{p-q}{pq}\right\}^n.$$

Ex. 31. In the last problem, if $p=60$, and $q=45$, shew that the population will be doubled in 125 years, nearly.

$$\text{Here } P_n = 2P = P\left\{1 + \frac{1}{180}\right\}^n, \text{ to find } n.$$

$$\therefore \left(\frac{181}{180}\right)^n = 2, \text{ or } n.\{\log 181 - \log 180\} = \log 2,$$

$$n\{2{\cdot}25768 - 2{\cdot}25527\} = 0{\cdot}30103,$$

$$\therefore n = \frac{0{\cdot}30103}{0{\cdot}00241} = 124{\cdot}9 = 125 \text{ nearly.}$$

CHANCES, AND LIFE ANNUITIES.

Ex. 1. What is the chance of drawing the four aces from a pack of cards in four successive trials?

The chance of drawing an ace the 1st time is $\frac{4}{52}$.

The chance of then drawing an ace the 2nd trial is $\frac{3}{51}$, since 3 aces only *may* remain, and there are 51 cards in all.

Similarly the chance of drawing an ace the 3rd trial is $\frac{2}{50}$.

And the chance of drawing an ace the 4th trial is $\frac{1}{49}$.

The compound chance $\therefore$ is $\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}$, or $\frac{1}{270725}$.

Ex. 2. There are 4 white balls, and 3 black, placed at random in a line, find the chance of the extreme balls being both black.

If 2 black balls be taken away, the 5 remaining balls can be placed in $\lfloor 5$ positions: if then the 2 excepted balls be placed one at each end of the line thus formed, we get $2 \times \lfloor 5$ favourable arrangements. Now 2 black balls can be taken out in 3 ways, so that in all there are $6 \times \lfloor 5$, or $\lfloor 6$, favourable arrangements. But the 7 balls admit of $\lfloor 7$ different arrangements,

$$\therefore \text{chance required} = \frac{\lfloor 6}{\lfloor 7} = \frac{1}{7}.$$

Ex. 3. Of two bags one contains 9 balls, and the other 6, and in each bag the balls are marked *a*, *b*, *c*, *d*, &c. If one be drawn from each bag, what is the chance that the two will have the same letter-mark?

They cannot be pairs, unless one of the first 6 letters be drawn from the first bag; the chance of this is $\frac{6}{9}$. Then the chance of drawing the same letter from the 2nd bag is $\frac{1}{6}$;

$$\therefore \text{ the chance of both happening} = \frac{6}{9} \times \frac{1}{6} = \frac{1}{9}.$$

Ex. 4. A plays one game with B, and another with C; the odds that he does not win both games are 4 to 1; and the odds that he beats B are 3 to 2. What are the odds in his game with C?

Let $\frac{x}{y}$ be the odds required; then the chance of his beating $B \times$ chance of his beating $C =$ chance of winning both games,

$$\text{or } \frac{3}{5} \times \frac{x}{x+y} = \frac{1}{5}.$$

$$\text{or } 3x = x + y, \quad 2x = y, \quad \frac{x}{y} = \frac{1}{2},$$

$\therefore$ the odds against him are 2 to 1.

Ex. 5. From a common pack of cards 12 are dealt to as many persons, one to each, the cards collected, shuffled, and the same repeated. What is the chance that a given person will on both occasions have the same card dealt to him? What is the chance that the two cards dealt to him will have the sum of their numbers equal to 3?

(1) The number 12 does not affect the chance, and the chance of having the same card the second time as the first is obviously $\frac{1}{52}$; since there is only 1 out of 52 cards which will serve the purpose.

(2) In order that the 2 cards may have the number of pips equal to 3, one must be an ace, and the other a deuce. Now the chance of one or other of these being dealt the first time is $\frac{8}{52}$; and when one of them has been dealt, then the chance of the other turning up the second time is $\frac{4}{52}$. Therefore the chance that these two independent events will both happen, or

$$\text{the chance required} = \frac{8}{52} \times \frac{4}{52} = \frac{2}{13} \times \frac{1}{13} = \frac{2}{169}.$$

Ex. 6. Two dice are placed together at random so as to form a parallelopiped: find the chance that two or more adjacent faces will have the same marks.

Let the dice be called A and B, and suppose each face in B to be applied successively to the ace of A, thus forming 6 parallelopipeds, each of which admits of four different arrangements with respect to the outer faces, and making 24 different arrangements in all. The same is true, if we begin with each of the other faces of A. Therefore the whole number of different arrangements is 6×24.

Now taking the first of these sets of 24, if to the ace of A, the ace of B be applied, of the 4 changes which can be made 2 will have similar adjacent faces of A and B. If any of the other 5 faces of B be applied, one only of the 4 changes can shew similar adjacent faces. Therefore there are 7 favourable cases out of 24. And similarly for each of the other sets of 24: that is, the number of favourable arrangements is 6×7,

$$\therefore \text{ chance required} = \frac{6\times 7}{6\times 24} = \frac{7}{24}.$$

Ex. 7. From a bag containing 2 guineas, 3 sovereigns, and 5 shillings, a person is allowed to draw 3 of them indiscriminately; what is the value of his expectation?

The different combinations of 3 are

Kind.	*Number.*	*Value in Shillings.*
2 G, and £1,	$1\times 3=3$,	$3\times 62=186$,
1 G, and 2£,	$2\times 3=6$,	$6\times 61=366$,
3£,	$1=1$,	$1\times 60=60$,
1 G, 1£, and 1*s*.,	$2\times 3\times 5=30$,	$30\times 42=1260$,
2 G, and 1*s*.,	$1\times 5=5$,	$5\times 43=215$,
1 G, and 2*s*.,	$2\times 10=20$,	$20\times 23=460$,
2£, and 1*s*.,	$3\times 5=15$,	$15\times 41=615$,
1£, and 2*s*.,	$3\times 10=30$,	$30\times 22=660$,
3*s*.,	10,	$10\times 3=30$,
	Total No. = 120	Total Value = 3852*s*.

$$\therefore \text{ Average value of } \textit{one} \text{ combination} = \frac{\overset{s.}{3852}}{120} = 32\tfrac{1}{10} \text{ shillings}.$$

Ex. 8. A shilling is thrown upon a chess-board, a square of which will just include 4 shillings, find the chance of its falling clear of a division.

Suppose the diameter of the shilling to be 1, then the side of the board will be 16, and the whole surface of the board will be 256; out of this there will be 64 squares, each area $=1$, in which the centre of the shilling must fall to miss a division, therefore the chance of falling clear of a division $=\frac{64}{256}=\frac{1}{4}$.

N.B. This is on the supposition that the shilling may lie with one half of its area over the edge of the board. But if the shilling must lie wholly on the board, this will exclude a border all the way round the board equal in breadth to the radius of the shilling and then the chance $=\frac{64}{225}$.

Ex. 9. Three men A, B, C, in succession throw a die, on condition that he who first throws an ace shall receive 1£; what are the values of their several expectations?

Since certainty is worth 1£,

A has	B has	C has
(1) Chance worth £$\frac{1}{6}$,	(2) Chance, after A's failing, worth £$\frac{5}{6^2}$,	(3) Chance, after A's and B's failing, worth £$\frac{5^2}{6^3}$,
(4) Chance next time, worth £$\frac{5^3}{6^4}$,	(5) Chance, next, worth £$\frac{5^4}{6^5}$,	&c.

and so on.

$$\text{Therefore value of } A\text{'s chance} = \left(\frac{1}{6}+\frac{5^3}{6^4}+\frac{5^6}{6^7}+\ldots\right)£,$$

$$=\frac{1}{6}\times\frac{1}{1-\left(\frac{5}{6}\right)^3}£=\frac{36}{91}£.$$

$$\text{Also } B\text{'s}=\frac{5}{6}\;A\text{'s}=\frac{5}{6}\times\frac{36}{91}=\frac{30}{91}£\;;\text{ and } C\text{'s}=\frac{5}{6}\;B\text{'s}=\frac{25}{91}£.$$

$$\therefore\; A\text{'s}=7s.\;10\tfrac{86}{91}d. \qquad B\text{'s}=6s.\;7\tfrac{11}{91}d. \qquad C\text{'s}=5s.\;5\tfrac{85}{91}d.$$

Ex. 10. There are 5 persons, out of which 4 are going to play at whist. They all cut, and the lowest sits out. What is the chance that two specified individuals will be partners?

Let A, B, C, D, E be the five persons, and A and B the two specified individuals. Now, 1st., the chance that both A and B play at all is the chance that any one of the three C, D, E, cuts

lowest, that is, $\frac{3}{5}$. Then, supposing A and B going to play, there are three different partners A *may* get, and only one, so that A and B may be partners; his chance therefore of getting B is $\frac{1}{3}$.

Hence the chance required, of both playing and playing as partners, is $\frac{3}{5} \times \frac{1}{3}$, or $\frac{1}{5}$.

Ex. 11. There is a lottery containing black and white balls, from each drawing of which it is as likely a black ball shall arise as a white one, what is the chance of drawing 11 balls all white?

The chance of drawing a white ball *each time* being $\frac{1}{2}$, the chance of doing so 11 times successively is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \text{\&c. to 11 factors} = \frac{1}{2^{11}} = \frac{1}{2048}.$$

Ex. 12. There is a lottery of 10 green, 12 white, and 14 red balls. Let two have been drawn, what is the probability that they will be green and white?

Two balls can be drawn from the whole in $\frac{36 \times 35}{1 \times 2}$, or 18×35 ways.

Each green ball can be combined with one white one in 12 different ways,

$\therefore$ the number of pairs of green and white $= 10 \times 12$ or 120.

$$\therefore \text{chance stated} = \frac{120}{18 \times 35} = \frac{4}{21};$$

$$\text{chance against it} = 1 - \frac{4}{21} = \frac{17}{21},$$

$\therefore$ odds against it are as 17 to 4.

Ex. 13. A die is thrown time after time: in how many times have we an even chance of throwing an ace?

The chance of failing in each throw singly is $\frac{5}{6}$, and in x throws successively is $\left(\frac{5}{6}\right)^x$; so that the question is to find the value of x, when

$$\left(\frac{5}{6}\right)^x = \frac{1}{2}.$$

By trial $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$, which is somewhat greater than $\frac{1}{2}$,

$\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$, ... less than $\frac{1}{2}$.

$\therefore$ No. required lies between 3 and 4.

Or, by the Tables, $x = \frac{\log 2}{\log 6 - \log 5} = \frac{0{\cdot}30103000}{0{\cdot}07918125} = 3{\cdot}8$, or $3\frac{4}{5}$, nearly, but something more.

Ex. 14. Two witnesses, on each of whom it is 3 to 1 that he speaks truth, agree in affirming that a certain event did happen, which of itself is equally likely to have happened or not. What is the chance that the event did happen?

The chance of each speaking truth being $\frac{3}{4}$, the chance of *both* speaking truth at the same time is $\frac{9}{16}$; of *both* lying is $\frac{1}{16}$; and the chance of one or other speaking truth and the other lying is excluded in this case, since they agree;

$\therefore$ chance of truth : chance of falsehood $:: \frac{9}{16} : \frac{1}{16} :: 9 : 1$.

Ex. 15. *A* speaks truth 3 times out of 4, *B* 4 times out of 5, *C* 6 times out of 7; what is the probability of the truth of what *A* and *B* agree in asserting, but which *C* denies?

After *A*'s assertion, $\frac{\text{the probability of the event}}{\text{improbability}} = \frac{3}{1}$.

Then *B*'s assertion increases this fraction in the ratio of 4 : 1, so that it becomes $\frac{12}{1}$.

Again, *C*'s denial diminishes the fraction in the ratio of 1 : 6, so that it becomes $\frac{2}{1}$. Therefore

Probability of the truth : improbability :: 2 : 1.

Ex. 16. Thirteen persons are required to take their places at a round table by lot; shew that it is 5 to 1 that two particular persons do not occupy contiguous seats.

Let one of the two have taken his place, then there are 12 places left, to each of which the other is equally liable, and only 2 places which will bring him next to his friend; $\therefore$ his chance of succeeding $= \frac{2}{12} = \frac{1}{6}$, and his chance of failing $= \frac{5}{6}$. $\therefore$ the odds against are 5 to 1.

Ex. 17. *P* bets *Q* £10 to £590 that three races will be won by the three horses *A*, *B*, *C*, against which the betting is 4 to 1, 3 to 1, and 2 to 1, respectively. The first race having been won by *A*, and it being known that the second race was won either by *B*, or by a horse *F* against which the betting was 6 to 1, find the value of *P*'s expectation.

$$\text{The chance of } B \text{ winning his race} = \frac{1}{4}.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots F \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{7},$$

$$\therefore B\text{'s chance} : F\text{'s} :: \frac{1}{4} : \frac{1}{7} :: 7 : 4,$$

$$\text{and } B\text{'s chance} : B\text{'s} + F\text{'s, or certainty} :: 7 : 11,$$

$$\therefore B\text{'s chance of winning} = \frac{7}{11}.$$

$$\text{Again, } C\text{'s chance of winning} = \frac{1}{3}.$$

$$\therefore \text{chance of } P \text{ winning his wager} = 1 \times \frac{7}{11} \times \frac{1}{3} = \frac{7}{33},$$

$$\text{and} \ldots\ldots\ldots\ldots \text{losing} \ldots\ldots\ldots\ldots = 1 - \frac{7}{33} = \frac{26}{33}.$$

$$\therefore \text{value of } P\text{'s expectation} = £\left(\frac{7}{33} \times 590 - \frac{26}{33} \times 10\right),$$

$$= £\frac{3870}{33} = £\frac{1290}{11},$$

$$= £117.\ 5s.\ 5\tfrac{5}{11}d.$$

Ex. 18. Supposing the House of Commons composed of m Tories and n Whigs, find the probability that a Committee of $p+q$ selected by ballot will consist of p Tories and q Whigs.

The N°. of different Committees of $(p+q)$ taken out of $(m+n)$ persons = the N°. of combinations of $(m+n)$ things taken $(p+q)$ together $= {}_{m+n}C_{p+q}$.

Again, the N°. of different parcels out of m Tories p together $= {}_mC_p$.

Also, ... n Whigs q together $= {}_nC_q$.

$\therefore$ the N°. of different ways, in which the required Committee can be formed $= {}_mC_p \times {}_nC_q$,

$$\therefore \text{chance} = \frac{{}_mC_p \times {}_nC_q}{{}_{m+n}C_{p+q}}.$$

Ex. 19. There are 3 balls in a bag, of which one is white and one black, and the third white or black; determine the chance of drawing two black ones, if two be taken.

If the unknown ball be white, then there is no way of drawing 2 *black* balls, but three ways of drawing 2 balls.

If the unknown ball be black, then there is one way of drawing 2 *black* balls, but three ways of drawing 2 balls.

So then of six ways equally probable only one is favourable;

$$\therefore \text{ chance} = \frac{1}{6}.$$

Ex. 20. In a lottery all the tickets are blanks but one; each person draws a ticket, and retains it. Shew that each person has an equal chance of drawing the prize.

Let n = the number of tickets; then

the chance of the 1st person drawing the prize $= \frac{1}{n}$,

and .... not .... $= 1 - \frac{1}{n}$ or $\frac{n-1}{n}$.

The 2nd person's chance of the prize = chance of 1st failing × chance of winning on that supposition

$$= \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}.$$

The 3rd person's chance of the prize

= chance of 1st and 2nd failing

× chance of winning in that case,

$$= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n},$$

and so on; shewing that the chances of all are equal.

Ex. 21. A collection is made of 10 letters taken at random from an alphabet consisting of 20 consonants and 5 vowels; what is the probability that it will contain three vowels and no more?

Whole number of combinations out of 25 letters 10 together

$$= \frac{25 \times 24 \times \ldots \times 16}{1 \times 2 \times \ldots \times 10}, \quad (a)$$

Number of different combinations out of 20 consonants 7 together $=\frac{20\times19\times\ldots\times14}{1\times2\times\ldots\times7}$,

.... 5 vowels 3 together $=\frac{5\times4\times3}{1\times2\times3}$,

$\therefore$ No. of ways in which 7 consonants and 3 vowels can be taken

$$=\frac{20\times19\times\ldots\times14}{1\times2\times\ldots\times7}\times\frac{5\times4\times3}{1\times2\times3}, \quad (\beta),$$

$$\therefore \text{probability required}=\frac{\beta}{\alpha}=\frac{3\times4\times5}{11\times23}=\frac{60}{253}.$$

Ex. 22. At the game of whist, what is the chance of dealing one ace and no more to a specified person? And what is the chance of dealing one ace to each person?

Let A, B, C, D be the four persons playing at whist: there are 24 permutations of the 4 aces, that is, 24 different ways in which they may succeed each other. We will first take one of these ways, viz.

Ace of Spades, of Hearts, of Diamonds, of Clubs,

and find the chance that A, B, C, D hold them respectively.

A's chance for the 1st $=\frac{13}{52}$. Then there are 51 cards in which the ace of Hearts must be (according to our supposition) and B holds 13 of them, $\therefore$ B's chance for 2nd ace $=\frac{13}{51}$. Similarly, C's chance for the ace of Diamonds $=\frac{13}{50}$; and D's chance for the ace of Clubs $=\frac{13}{49}$;

$$\therefore \text{chance of all this together}=\frac{13}{52}\times\frac{13}{51}\times\frac{13}{50}\times\frac{13}{49}.$$

Hence, the same being done for each of the other permutations of the 4 aces,

$$\text{the chance required}=24\times\frac{13}{52}\times\frac{13}{51}\times\frac{13}{50}\times\frac{13}{49}=\frac{2197}{20825}, \text{ or } \frac{1}{9} \text{ nearly.}$$

Again, let A be the specified person who is to have one and only one ace; whole N°. of different hands A can have $={}_{52}C_{13}$. And the N°. of these containing one and only one ace $={}_{48}C_{12}\times4$;

$$\therefore \text{chance required}=\frac{{}_{48}C_{12}}{{}_{52}C_{13}}\times4=\frac{39\times38\times37}{51\times50\times49}=\frac{9139}{20825}=\frac{32}{73} \text{ nearly.}$$

Ex. 23. There are 2 bags, each containing 4 white and 4 black balls. Four are taken at a venture from one of them, and transferred to the other. Then 8 being drawn from the latter, 6 of them prove white and 2 black: what is the chance that, if another be drawn, it will be white?

Since it is known, that 2 white balls were drawn, the *first* drawing may consist of one or other of the following sets:

$$\left.\begin{array}{ll} \text{2 white, 2 black, the chance of which is} & \dfrac{{}_4C_2\times{}_4C_2}{{}_8C_4}=\dfrac{36}{70} \\ \text{3 white, 1 black, ...} & \dfrac{{}_4C_3\times{}_4C_1}{{}_8C_4}=\dfrac{16}{70} \\ \text{4 white, none black ....} & \dfrac{{}_4C_4}{{}_8C_4}=\dfrac{1}{70} \end{array}\right\} \text{(A)};$$

and then, before the drawing, the other bag would contain

6 white and 6 black, 7 white and 5 black, or 8 white and 4 black. The chances of drawing 6 white and 2 black from these are respectively

$$\frac{{}_6C_6\times{}_6C_2}{{}_{12}C_8},\ \frac{{}_7C_6\times{}_5C_2}{{}_{12}C_8},\ \frac{{}_8C_6\times{}_4C_2}{{}_{12}C_8};\ \text{or}\ \frac{15}{495},\ \frac{70}{495},\ \frac{168}{495};$$

$\therefore$ the chances for the observed event from the separate cases of (A) are respectively

$$\frac{36}{70}\times\frac{15}{495},\ \frac{16}{70}\times\frac{70}{495},\ \frac{1}{70}\times\frac{168}{495}.$$

There are now remaining 4 black, 1 white and 3 black, 2 white and 2 black. And the chances of the next ball being white are, therefore, 0, $\frac{1}{4}$, $\frac{1}{2}$, respectively.

$$\text{Hence, required chance}=\frac{36\times15\times0+16\times70\times\frac{1}{4}+1\times168\times\frac{1}{2}}{36\times15+16\times70+1\times168};$$

$$=\frac{91}{457}.$$

Ex. 24. At the game of whist what is the chance of the dealer and his partner holding the 4 honours?

1st. The chance of the dealer turning up an honour $=\frac{4}{13}$, and the chance of his not turning up an honour $=\frac{9}{13}$. In both cases the dealer and his partner have 25 cards out of the remaining 51, which they may have in ${}_{51}C_{25}$ different ways. Also the number of these hands which contain the 3 honours will be ${}_{48}C_{22}$. Hence, if an honour be turned up, the chance then of the other 3 honours being with the dealer and his partner

$$=\frac{{}_{48}C_{22}}{{}_{51}C_{25}}=\frac{25\times24\times23}{51\times50\times49}=\frac{92}{833},$$

$$\therefore \text{ the compound chance of these 2 events} = \frac{4}{13} \times \frac{92}{833}.$$

But if the dealer does not turn up an honour, the chance of the 4 honours being given to the dealer and his partner out of the remaining cards

$$= \frac{{}_{47}C_{21}}{{}_{51}C_{25}} = \frac{25 \times 24 \times 23 \times 22}{51 \times 50 \times 49 \times 48} = \frac{253}{4998};$$

$$\therefore \text{ the compound chance in this case} = \frac{9}{13} \times \frac{253}{4998}.$$

Hence the *whole* chance, taking both cases together, or the chance required

$$= \frac{4}{13} \times \frac{92}{833} + \frac{9}{13} \times \frac{253}{4998} = \frac{115}{1666}, \text{ or } \frac{2}{29} \text{ nearly.}$$

Ex. 25. In dealing a pack of cards, what is the chance that all the hearts will be found in the 20 cards first dealt, 1st without regard to order, 2ndly in the order of their value?

1st. Seven cards can be taken from 39, the residue of the pack with the hearts taken out, in $\dfrac{39 \times 38 \times 37 \times 36 \times 35 \times 34 \times 33}{\underline{|7}}$ ways. Unite to one of these the hearts, and we have a group of 20 cards containing the hearts. These can be permuted in $\underline{|20}$ ways, and so give $\dfrac{\underline{|39}}{\underline{|7} \times \underline{|32}} \times \underline{|20}$ ways of dealing the first 20 cards with all the hearts in them. Moreover the remaining 32 can be permuted in dealing in $\underline{|32}$ ways. So the number of ways of *successful* dealing is

$$\frac{\underline{|39}}{\underline{|7} \times \underline{|32}} \times \underline{|20} \times \underline{|32},$$

$$\text{or } \frac{\underline{|39} \times \underline{|20}}{\underline{|7}}.$$

But the number of ways of dealing in any manner is $\underline{|52}$;

$$\therefore \text{ chance} = \frac{\underline{|39} \times \underline{|20}}{\underline{|7} \times \underline{|52}}.$$

2nd. Let P be the number of ways of dealing the pack, with all the hearts in the first 20 cards, and in the order of their value. Then any one of these dealings admits of $\underline{|13}$ different dealings by permuting the hearts, $\therefore P \times \underline{|13} =$ the whole number of ways of dealing so as to have all the hearts in the first 20 cards, but in any order, $= \dfrac{\underline{|39} \times \underline{|20}}{\underline{|7}}$ by the 1st case.

$$\therefore\ P = \frac{\underline{|39} \times \underline{|20}}{\underline{|7} \times \underline{|13}},$$

$$\therefore \text{ chance required} = \frac{\underline{|39} \times \underline{|20}}{\underline{|7} \times \underline{|13} \times \underline{|52}}.$$

Ex. 26. A person puts his hand into a bag containing 12 balls, draws out a certain number at random, and transfers them without examination to a second empty bag. He then puts his hand into this second bag, and draws out a certain number in the same way as before. Shew that the odds in favour of his drawing out an odd number from the second bag are nearly 341 to 340.

Suppose that he has taken n balls out of the first bag.

Then from these n he can draw an odd number in 2^{n-1} ways, and an odd *or* even number in 2^n-1 ways: for the *total* number of combinations of n things is 2^n-1, and the number of *odd* combinations is 2^{n-1} *.

Therefore taking the different cases as n ranges from 1 to 12, we have the chance of drawing an odd number

$$= \frac{1+2+2^2+\ldots+2^{11}}{1+2+2^2+\ldots+2^{12}-13},$$

$$= \frac{2^{12}-1}{2^{13}-14} = \frac{4095}{8178}.$$

And the chance against it $= 1 - \dfrac{4095}{8178} = \dfrac{4083}{8178}$,

$$\therefore \text{ the odds in favour} = \frac{4095}{4083} = 1 + \frac{1}{340+\&c.},$$

$$= \frac{341}{340}, \text{ nearly.}$$

Ex. 27. Supposing it an even chance that, on A aged 46 marrying B aged 36, they will live together x years; find x, when, according to De Moivre's hypothesis, of 86 persons born together, one dies annually until all are extinct.

By Art. 455,

A's chance of being alive at the end of x years is $\dfrac{40-x}{40}$,

B's .... is $\dfrac{50-x}{50}$,

* This is easily seen by making $x=1$ in the following equality,

$$\frac{(1+x)^n-(1-x)^n}{2} = nx + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} + \&c.$$

$$\therefore \text{ chance of } \textit{both} \text{ being alive} = \frac{40-x}{40} \times \frac{50-x}{50},$$

$$\therefore \text{ by the question, } \frac{40-x}{40} \cdot \frac{50-x}{50} = \frac{1}{2},$$

$$2000 - 90x + x^2 = 1000,$$

$$x^2 - 90x + (45)^2 = 1025,$$

$$x - 45 = 32, \text{ very nearly,}$$

$$\therefore x = 13, \ldots\ldots\ldots\ldots\ldots$$

Ex. 28. A person 35 years of age wishes to buy an annuity for what may happen to remain of his life after 50 years of age. What is the Present Value of the annuity, reckoning interest at 4 per cent., and using Dr Halley's Table, and De Moivre's hypothesis?

The Present Value of such an Annuity *to commence immediately*, is found by substituting in Art. 455, for n and R, 51 and 1·04*.

The Present Value for the first 15 years is found from Art. 418, by putting 15 for n, and 1·04 for R.

The difference of these two results will be the *Present Value* of the annuity required, if it were certain that the person would live to receive it at all, that is, to the age of 50.

The chance of this, by the Table, is $\frac{346}{490}$.

$$\therefore \text{ Value required} = \frac{346}{490} \times \text{difference above mentioned,}$$

$$= 4{\cdot}44, \text{ or } 4\tfrac{1}{2} \text{ nearly, years purchase.}$$

Ex. 29. An annuity of £10 for the life of a person now 30 years old, is to commence† at the end of 11 years, if another person now 40 should then be dead. Find the Present Value of the annuity, reckoning interest at 4 per cent., and using Dr Halley's Table.

The Present Value of an annuity of £10, for the life of a person now 30, and to commence after 11 years, found as in the last example, is £69·43; and the probability of a person now 40 dying in 11 years is $1 - \frac{335}{445}$, or $\frac{110}{445}$.

* The numerical calculations in this and the three following examples are similar to those in Ex. Art. 422.

† That is, his *title* to the annuity, the first *payment* being made a year afterwards.

$\therefore$ the Present Value required $= \frac{110}{445} \times 69 \cdot 43 = £17 \cdot 16$.

Ex. 30. An estate, or annuity of £10 *for ever,* will be lost to the heirs of a person now in possession and aged 34, if his life should fail before the expiration of 11 years. What ought he to give for the *assurance* of it for this term, reckoning interest at 4 per cent.?

The estate, or annuity, is obviously worth £250 of *ready money* on any of the days on which the rent or annuity has been paid. It is evident then that to prevent any possible loss to his heirs, the man must insure immediately after rent day for a term of 11 years to the amount of £260, payable on the next rent-day after his decease.

The *present* probability of his dying in each of the years is $\frac{1}{52}$, according to De Moivre's hypothesis,

$\therefore$ the Present Value of £260 to be paid at his death, if it occur within 11 years,

$$= \frac{260}{52} \times \left\{ \frac{1}{1 \cdot 04} + \frac{1}{(1 \cdot 04)^2} + \frac{1}{(1 \cdot 04)^3} + \text{\&c. to 11 terms} \right\},$$

$$= \frac{260}{52} \times \frac{1 - (1 \cdot 04)^{-11}}{\cdot 04},$$

$$= \frac{260}{52} \times \frac{\cdot 3504}{\cdot 04}, \left\{ \begin{array}{l} \text{using the Log}^{c}\text{. Tables to calculate} \\ 1 - (1 \cdot 04)^{-11}. \end{array} \right.$$

$$= \frac{65}{13} \times 8 \cdot 76,$$

$$= £43 \cdot 8, \text{ or } £43.\ 16s.$$

Ex. 31. A person now 40 is willing to pay £200 down, *besides* an annual payment for 10 years, to entitle him to a life-annuity of £44 after he attains the age of 50. What ought the annual payment to be, reckoning interest at 4 per cent., and using Dr Halley's Table?

1st. Find the Present Value of an annuity of £1 for life to a person now 40, but not to commence until he is 50; this will be found to be £5·953,

$\therefore$ Present Value of $£44 = 44 \times 5 \cdot 953 = £261 \cdot 932$.

The required *annual* payment, then, must be equivalent to £61·932 down; because he only engages to pay £200 instead of £261·932, which is the real value of his annuity.

Again, the Present Value of an annual payment of £1 to continue for life and to commence immediately is (by De Moivre's hypothesis and Art. 456)

$$\frac{1}{\cdot 04}-\frac{1\cdot 04}{46}\times\frac{1-(1\cdot 04)^{-46}}{(\cdot 04)^2},$$

$$=25-\frac{1\cdot 04}{46}\times\frac{\cdot 83538}{(\cdot 04)^2}.$$

$$=£13\cdot 196.$$

$\therefore$ Present Value of £1 for 10 years $=13\cdot 196-5\cdot 953=£7\cdot 243$,

$$\therefore \text{the annual payment required}=\frac{61\cdot 932}{7\cdot 243}=£8\cdot 55.$$

MISCELLANEOUS EXAMPLES.

(FIRST SERIES.)

Ex. 1. Find the Greatest Common Measure of

$$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz,$$

$$\text{and } (ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz.$$

1st quantity

$$=(ax+by)\{ax+by-(ax+az-bx-bz)\}+(a-b)^2xz,$$

$$=(ax+by)b(x+y)+(ax+by)(bz-az)+(a-b)^2xz,$$

$$=(ax+by)b(x+y)+abxz-a^2xz+b^2yz-abyz+a^2xz-2abxz+b^2xz,$$

$$=(ax+by)b(x+y)+b^2z(x+y)-abz(x+y),$$

$$=b(x+y)\{ax+by+bz-az\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The 2nd quantity differs from the first only by having $-b$ instead of b, therefore 2nd quantity

$$=-b(x+y)\{ax-by-bz-az\},$$

$$=b(x+y)\{by-ax+bz+az\}\ldots\ldots\ldots\ldots(2).$$

Hence, the latter factors in (1) and (2) having no Common Measure greater than 1, the Greatest Common Measure required is $b(x+y)$.

Ex. 9. Shew that $x^n - na^{n-1}x + (n-1)a^n$ is divisible by $(x-a)^2$, if n be a whole number.

$$x^n - na^{n-1}x + (n-1)a^n = x^n - a^n - na^{n-1}(x-a),$$

$$\div \text{ by } x-a \quad = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1} - na^{n-1},$$

$$= (x^{n-1} - a^{n-1}) + a(x^{n-2} - a^{n-2}) + a^2(x^{n-3} - a^{n-3}) + \&c.$$

and *each* of the quantities enclosed within the n brackets is divisible by $x-a$, (Art. 99, Ex. 6), therefore the proposed quantity is divisible by $(x-a)^2$.

Ex. 10. Shew that $x^p - y^n$ is divisible by $x + y^{\frac{n}{p}}$, when p is an even number.

Let $p = 2m$, an even number, then

$$\frac{x^p - y^n}{x + y^{\frac{n}{p}}} = \frac{x^{2m} - y^n}{x + y^{\frac{n}{2m}}} = \frac{X^2 - Y^2}{X^{\frac{1}{m}} + Y^{\frac{1}{m}}}, \text{ writing } X \text{ for } x^m, \text{ and } Y \text{ for } y^{\frac{n}{2}},$$

$$= \frac{X^2 - Y^2}{X \pm Y} \cdot \{X^{1-\frac{1}{m}} - X^{1-\frac{2}{m}}Y + \&c.\} \text{ (Art. 177)}$$

$$= (X \mp Y)\{X^{1-\frac{1}{m}} - X^{1-\frac{2}{m}}Y + \&c.\},$$

that is, $x^p - y^n$ is divisible by $x + y^{\frac{n}{p}}$.

Ex. 11. If N and n be nearly equal to each other, shew that

$$\sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{1}{4} \cdot \frac{N+n}{n}, \text{ very nearly;}$$

and that, if $\dfrac{N}{N+n}$ and $\dfrac{1}{4} \cdot \dfrac{N+n}{n}$ have their first p decimal places the same, this approximation to $\sqrt{\dfrac{N}{n}}$ may be relied on to $2p$ decimals at least.

(1) Let $N = m+x$, and $n = m-x$, x being very small,

$$\text{then } \sqrt{\frac{N}{n}} = \left(\frac{m+x}{m-x}\right)^{\frac{1}{2}} = \frac{(m^2 - x^2)^{\frac{1}{2}}}{m-x},$$

$$= \frac{m}{m-x}\left\{1 - \frac{x^2}{2m^2}\right\} \text{ nearly,}$$

$$=\frac{2m^2-x^2}{2m(m-x)}=\frac{m^2-x^2}{2m(m-x)}+\frac{m}{2(m-x)},$$

$$=\frac{m+x}{2m}+\frac{1}{4}\cdot\frac{2m}{m-x},$$

$$=\frac{N}{N+n}+\frac{1}{4}\cdot\frac{N+n}{n}.$$

(2) Also $\frac{1}{4}\cdot\frac{N+n}{n}-\frac{N}{N+n}$, or $\frac{x^2}{2m(m-x)}<\frac{1}{10^p}$, by supposition, $\therefore$ *à fortiori* $\frac{x^2}{2m^2}<\frac{1}{10^p}$, and also $\frac{x^2}{4m^2}<\frac{1}{10^p}$. But the 1st term neglected in the series for $\left(\frac{N}{n}\right)^{\frac{1}{2}}$ is $\frac{x^4}{8m^3(m-x)}$, which $<\frac{x^4}{8m^4}<\frac{x^2}{2m^2}\times\frac{x^2}{4m^2}<\frac{1}{10^p}\cdot\frac{1}{10^p}<\frac{1}{10^{2p}}$. Similarly the next term $<\frac{1}{10^{3p}}$; and so on. Therefore $\frac{N}{N+n}+\frac{1}{4}\cdot\frac{N+n}{n}$ is the correct value of $\sqrt{\frac{N}{n}}$ to $2p$ decimal places at least.

Ex. 12. Find the value of x, which, when z is indefinitely increased, makes $(4x+1)(2z+1)^2=5(3x+1)(z+3)^2$; also find the values of x and y which make $\frac{2z^2+(x-a)z+2b(x-2c)}{3z^2+(y-b)z+3a(y-3c)}$ independent of z.

(1) When z is very large, all powers of z below the highest may be neglected,

$$\therefore\ (4x+1)4z^2=5(3x+1)z^2,$$

$$16x+4=15x+5,$$

$$\therefore\ x=1.$$

(2) Suppose the given fraction $=m$ for all values of z; then

$$2z^2+(x-a)z+2b(x-2c)=3mz^2+(y-b)mz+3am(y-3c),$$

$$\text{or}\ \ (2-3m)z^2+\{(x-a)-m(y-b)\}z+2b(x-2c)-3am(y-3c)=0,$$

an equation which can be independent of z only when the coefficients of z^2 and z are separately equal to 0.

$$\left.\begin{aligned}\therefore\ 2-3m=0,\ldots\ldots\ldots\ldots(1)\\ x-a-m(y-b)=0,\ldots\ldots(2)\\ 2b(x-2c)-3am(y-3c)=0\ldots(3)\end{aligned}\right\}$$

From (1) and (2) $\quad 3x-3a-2(y-b)=0,$

$$\left.\begin{aligned} 3x-2y &= 3a-2b, \\ bx-ay &= 2bc-3ac, \end{aligned}\right\}$$

Again, from (3),

from which we get $x=a+2c$, and $y=b+3c$.

Ex. 13. Shew that $x^4+px^3+qx^2+rx+s$ can be resolved into two rational quadratic factors, if $-s$ be a perfect square and equal to $\dfrac{r^2}{p^2-4q}$.

Since $-s$ is a perfect square, let it be n^2, and assume

$$\begin{aligned} x^4+px^3+qx^2+rx-n^2 &= (x^2+Ax+n)(x^2+Bx-n) \\ &= x^4+(A+B)x^3+ABx^2-n(A-B)x-n^2, \end{aligned}$$

$$\left.\begin{aligned} \therefore\ A+B &= p, \\ AB &= q, \\ A-B &= \frac{-r}{n}, \end{aligned}\right\} \quad \begin{aligned} &\text{and } p^2-4q=(A-B)^2=\frac{r^2}{n^2}, \\ &\therefore\ \frac{r^2}{p^2-4q}=n^2=-s. \end{aligned}$$

Ex. $\quad x^4-6x^3+5x^2+8x-4=0.$

$$\text{Here}\quad \left.\begin{aligned} A+B &= -6, \\ A-B &= -4, \end{aligned}\right\} \qquad \left.\begin{aligned} \therefore\ A &= -5, \\ \text{and } B &= -1, \end{aligned}\right\}$$

and the factors are x^2-5x+2, x^2-x-2;

and $x^2-5x+2=0$, $x^2-x-2=0$, will give the four values of x.

Ex. 14. Prove that the square root of $\left(x-\dfrac{1}{x}\right)^2+\left(y-\dfrac{1}{y}\right)^2$ is a *rational* function of a and b, if $x=\dfrac{1}{4}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$, and $y=\dfrac{ax-h}{a-bx}$.

Let $b=at$, then $x=\dfrac{1+t^2}{4t}$, and $y=\dfrac{x-t}{1-tx}$,

$$\begin{aligned} \therefore\ y-\frac{1}{y} &= \frac{x-t}{1-tx}-\frac{1-tx}{x-t}, \\ &= \frac{x^2+t^2-1-t^2x^2}{(1-tx)(x-t)}, \\ &= \frac{(1-t^2)\left(x-\dfrac{1}{x}\right)}{(1-tx)\left(1-\dfrac{t}{x}\right)}. \end{aligned}$$

But $$(1-tx)\left(1-\frac{t}{x}\right)=1+t^2-\left(x+\frac{1}{x}\right)t,$$

$$=1+t^2-\frac{(1+t^2)^2+(4t)^2}{4(1+t^2)},$$

$$=\frac{4(1-t^2)^2-(1+t^2)^2}{4(1+t^2)},$$

$$\therefore\ y-\frac{1}{y}=\frac{4(1-t^2)(1+t^2)}{4(1-t^2)^2-(1+t^2)^2}\left(x-\frac{1}{x}\right),$$

$$\therefore\ \left(x-\frac{1}{x}\right)^2+\left(y-\frac{1}{y}\right)^2=\left\{1+\left[\frac{4(1-t^2)(1+t^2)}{4(1-t^2)^2-(1+t^2)^2}\right]^2\right\}\left(x-\frac{1}{x}\right)^2,$$

$$=\left\{\frac{4(1-t^2)^2+(1+t^2)^2}{4(1-t^2)^2-(1+t^2)^2}\right\}^2\left(x-\frac{1}{x}\right)^2,$$

$$\therefore\ \text{square root required}=\frac{4(1-t^2)^2+(1+t^2)^2}{4(1-t^2)^2-(1+t^2)^2}\left(x-\frac{1}{x}\right),$$

= a *rational* quantity, and a function of a and b, when $\frac{b}{a}$ is put for t, and $\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}\right)$ for x.

Ex. 15. Shew in what cases $ay^2+bxy+cx^2+dy+ex+f$ can be resolved into rational factors of the first degree.

It is easily shewn that the factors of Ay^2+By+C are rational when B^2-4AC is a square, and equal when it is zero. Therefore the factors of

$$ay^2+(bx+d)y+cx^2+ex+f$$

are rational when

$$(bx+d)^2-4a(cx^2+ex+f) \text{ is a square,}$$

$$\text{or } (b^2-4ac)x^2+(2bd-4ae)x+d^2-4af \text{ is a square,}$$

$$\text{that is, when } (bd-2ae)^2=(b^2-4ac)(d^2-4af), \text{ Art. 152,}$$

$$\text{or } ae^2+cd^2+fb^2=bde+4acf.$$

Ex. 16. A number consists of n figures, find the number of figures in its r^{th} root.

Let x be the number of figures in a, the r^{th} root of the given number; then

$$a \text{ is less than } 10^x, \text{ but not less than } 10^{x-1},$$

$\therefore a^r$ is less than 10^{rx}, but not less than 10^{rx-r}.

$\therefore n$ is not greater than rx, but not less than $rx-r+1$,

$\therefore x$ is not less than $\frac{n}{r}$, but not greater than $\frac{n+r-1}{r}$.

It follows that, if qr be the greatest multiple of r in $n+r-1$, $x=q$.

Ex. 17. In the year 1843 January 1 was a Sunday; when will this happen again?

Since an ordinary year consists of 365 days, or $52\times7+1$, days, and a leap-year of $52\times7+2$ days, it is clear that Jan. 1 must fall 1 day later in the week in 1844 than in 1843, 2 days later in 1845 than in 1844; and so on. Therefore its distance from Sunday in any year will be measured in days by the sum of as many terms of the series,

$$1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, \&c.$$

as expresses the number of years past 1843. Let $1843+x$ be the year required; then Jan. 1 will be a Sunday in $1843+x$, whenever the sum of x terms of the above series is divisible by 7. Now this is the case when $x=11$, $11+6$, $11+6+5$, $11+6+5+6$, &c. Therefore the years required are 1854, 1860, 1865, 1871, &c.

Ex. 18. What day of the week was Sept. 14, 1752, given that the same day of the month A.D. 1846 was a Monday?

From Sept. 14, 1752 to Sept. 14, 1846, 94 years have elapsed, of which 22 were leap-years. But Sept. 14, 1846 was Monday. Therefore, x being the day of the week on which Sept. 14, 1752 fell, 2 is the remainder when $94+22+x$ is divided by 7.

$$\text{Now } 94+22+x=7\times17+x-3,$$

$$\therefore x-3=2,$$

$$\therefore x=5,$$

$\therefore$ Sept. 14, 1752, was the 5th day of the week, or Thursday.

Ex. 19. How often are there 5 Sundays in February, and when does this happen in the present century after the year 1844, the 1st day of that year being Monday?

Jan. 1, 1844 was the 2nd day of the week, and Feb. 1 was therefore the 5th day. And, since 1844 was leap year, $1844+4x$ is leap year. Let it be one of those in which February has 5 Sundays, i.e. in which Feb. 1 is a Sunday. Then

$$4x+x+5 \text{ is of the form } 7n+1,$$
$$\text{or } 5x+4 \text{ is divisible by } 7,$$
$$5(x-2)+14 \ldots\ldots\ldots\ldots\ 7,$$
$$\therefore\ x-2=7p,$$
$x=7p+2$, where p is any positive whole number.

$\therefore$ $1844+28p+8$ is the year in which February has 5 Sundays.

Hence, making $p=0$, 1852 is the next such year, after 1844, and the interval between two of them is 28 years, until we reach the year 1900, when the omission of a leap-year will alter the case.

Ex. 20. Shew that if q be the integral part of $\frac{p}{4}$, and r the remainder when $p+q+4$ is divided by 7, then in the p^{th} year of this century Advent Sunday is generally $7-r$ days after Nov. 27; but, when $r=0$, it falls on that day.

An ordinary year exceeds 52 weeks by 1 day; and a leap-year by 2. If then q is the integral part of $\frac{p}{4}$, so that q will be the number of leap-years in p years; and supposing Nov. 27 to be the $(1+x)$th day of the week in the year 1800, then in the p^{th} year of this century, Nov. 27 will be the $(1+r)$th day of the week, if r be the remainder after dividing $p+q+x$ by 7.

But for 1846, $p=46$, $q=11$, and $p+q=57$; and (Nov. 27 being a Friday) $1+r=6$. Therefore 5 is the remainder (r) when $57+x$ is divided by 7; consequently $x=4$.

Also Advent Sunday, being the nearest Sunday to Nov. 30, falls on Nov. 27 if that day be a Sunday, i.e. if $r=0$; but in all other cases it will obviously fall $(7-r)$ days after Nov. 27.

Ex. 23. A steam vessel leaves Oban for Staffa with a supply of whiskey b above proof, (which is assumed to mean that $a+b$ gallons of spirit are mixed with c gallons of water) sufficient for two days' consumption, provided it receive no addition to its crew. On arriving at Tobermory m of its passengers remain behind, but by reason of contrary winds its progress to Staffa the following morning is retarded, so that on its return to Oban it is with difficulty enabled to reach Iona by midnight. It here receives n additional passengers, and also p gallons of whiskey d above proof. On an average each passenger dilutes his whiskey with water till it is e below proof, and consumes q pints of the mixture daily. The vessel arrives at Oban on the evening of the third day after its departure, by which time the supplies of whiskey

are both exhausted. Required the number of passengers on board when it left Oban, and the number of gallons of whiskey in the first supply.

Let x be the number of passengers at first; then, since each passenger consumes q pints daily, e below proof, and the supply is for two days, $2qx$ = number of pints, e below proof, in the first supply.

Let y = number of gallons, b above proof, in the first supply; then

$$y : \text{gallons of } spirit \text{ in the first supply} :: a+b+c : a+b,$$

$$\therefore \text{ gallons of } spirit \text{ in the first supply} = \frac{a+b}{a+b+c}.y.$$

Also $2qx$: *pints* of spirit in first supply :: $a-e+c : a-e$,

$$\therefore\ pints \text{ of spirit} = \frac{a-e}{a-e+c}.2qx,$$

$$\text{and } \therefore\ \frac{a+b}{a+b+c}.4y = \frac{a-e}{a-e+c}.qx \text{..................}(1).$$

$x-m$ = number of passengers on leaving Tobermory,

and each man consumes $\frac{q}{8}.\frac{a-e}{a-e+c}$ gallons of *spirit* daily,

$x.\frac{q}{8}.\frac{a-e}{a-e+c}$ = gallons of *spirit* consumed from Oban to Tobermory, that is, the 1st day.

$\therefore\ (x-m).\frac{q}{8}.\frac{a-e}{a-e+c}$ = gallons of *spirit* consumed from Tobermory to Iona, the 2nd day.

and $(x-m+n).\frac{q}{8}.\frac{a-e}{a-e+c}$ = gallons of *spirit* from Iona to Staffa, the 3rd day.

$\therefore\ \frac{q}{8}.\frac{a-e}{a-e+c}\{x+x-m+x-m+n\}$ = the whole quantity of spirit consumed,

$$=\frac{q}{4}.x.\frac{a-e}{a-e+c}+p.\frac{a+d}{a+d+c} \text{..................}(2).$$

$$x.\frac{q}{8}.\frac{a-e}{a-e+c}=(2m-n)\frac{q}{8}.\frac{a-e}{a-e+c}+p.\frac{a+d}{a+d+c},$$

$$\therefore\ x=2m-n+\frac{8p}{q}.\frac{a+d}{a-e}.\frac{a-e+c}{a+d+c}.$$

Hence also $y=\dfrac{a+b+c}{a+b}.x.\dfrac{q}{4}.\dfrac{a-e}{a-e+c}$,

$$=(2m-n)\frac{q}{4}.\frac{a-e}{a+b}.\frac{a+b+c}{a-e+c}+2p.\frac{a+d}{a+b}.\frac{a+b+c}{a+d+c}.$$

MISCELLANEOUS EXAMPLES.

(SECOND SERIES.)

Ex. 1. $2^{x+1}+4^x=80$, find x.

$$2\times2^x+2^{2x}=80,$$
$$2^{2x}+2\times2^x+1=81,$$
$$2^x+1=9,$$
$$2^x=8=2^3,$$
$$\therefore\ x=3.$$

Ex. 2. $x^3+3x=a^3-\dfrac{1}{a^3}$, find x.

Let $a-\dfrac{1}{a}=p$, then $a^3-\dfrac{1}{a^3}-3\left(a-\dfrac{1}{a}\right)=p^3$,

$\therefore\ x^3+3x=p^3+3p$,

$x^3-p^3+3(x-p)=0$, $\therefore\ x-p=0$, or $x=p\ldots(1)$,

and $x^2+px+p^2+3=0$,

$$x^2+px+\frac{p^2}{4}=-\frac{3p^2}{4}-3,$$

$$\therefore\ x=\frac{1}{2}\{-p\pm\sqrt{-3}.\sqrt{p^2+4}\},\ \text{where } p=a-\frac{1}{a},$$

$$=\frac{1}{2}\left\{-\left(a-\frac{1}{a}\right)\pm\sqrt{-3}\left(a+\frac{1}{a}\right)\right\}.$$

Ex. 3. $\left.\begin{array}{l} 7^{\left(\frac{x}{2}+\frac{y}{3}\right)}=2401, \\ 6^{\left(\frac{x}{4}+\frac{y}{2}\right)}=1296, \end{array}\right\}$ find x and y.

Since $2401=7^4$, and $1296=6^4$,

$$\left.\begin{aligned}\therefore\ \frac{x}{2}+\frac{y}{3}&=4,\\ \frac{x}{4}+\frac{y}{2}&=4,\end{aligned}\right\}\text{ two simple equations, to find } x \text{ and } y.$$

Ex. 4. $(x-3)(x-4)(x-5)(x-6)=1\times2\times3\times4$, find x,

$$(x-3)(x-6)(x-4)(x-5)=24,$$

$$(x^2-9x+18)(x^2-9x+20)=24,$$

write y for x^2-9x, $(y+18)(y+20)=24$,

$$y^2+38y=-336,$$

from which $y=-14$, or -24,

that is, $x^2-9x=-14$, or -24,

from which $x=7$, or 2, or $\frac{1}{2}(9\pm\sqrt{-15})$.

Ex. 5. $x^4+ax^3+bx^2+cx+\frac{c^2}{a^2}=0$, find x.

$$\left(x^2+\frac{c}{a}\right)^2+ax\left(x^2+\frac{c}{a}\right)+\left(b-\frac{2c}{a}\right)x^2=0,$$

$$\left(x^2+\frac{c}{a}\right)^2+ax\left(x^2+\frac{c}{a}\right)+\frac{a^2x^2}{4}=\left(\frac{a^2}{4}-b+\frac{2c}{a}\right)x^2,$$

$$x^2+\frac{c}{a}+\frac{ax}{2}=\sqrt{\frac{a^2}{4}-b+\frac{2c}{a}}\,.\,x,$$

a quadratic equation, to find x.

Ex. 6. Reduce to simplest form

$$\frac{(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ac)(a+bc)}{1-a^2-b^2-c^2-2abc}.$$

$$\text{Fraction}=\frac{\text{Numerator}}{(1-b^2)(1-c^2)-(a+bc)^2},$$

$$=1-a^2+(a+bc)\,.\,\frac{(1-a^2)(a+bc)-(c+ab)(b+ac)}{(1-b^2)(1-c^2)-(a+bc)^2}$$

$$=1-a^2+(a+bc).\frac{a+bc-a^2(a+bc)-bc-ab^2-ac^2-a^2bc}{(1-b^2)(1-c^2)-(a+bc)^2},$$

$$=1-a^2+(a+bc)a.\frac{1-a^2-b^2-c^2-2abc}{1-a^2-b^2-c^2-2abc},$$

$$=1+abc.$$

Ex. 7. Find x when

$$(x+a)(x+b)(x+c)(x+d)-(x-a)(x-b)(x-c)(x-d)$$
$$=(a+b+c+d)\{(x+a)(x+b)(x+c)+(x-a)(x-b)(x-c)\}$$

is an '*equation*'; and shew that there are 3, and only 3, relations between a, b, c, any one of which will cause it to be an '*identity*'.

$$(x+a)(x+b)(x+c)\{(x+d)-(a+b+c+d)\}$$
$$=(x-a)(x-b)(x-c)\{a+b+c+d+(x-d)\},$$

$$(x+a)(x+b)(x+c)\{x-(a+b+c)\}=(x-a)(x-b)(x-c)\{x+a+b+c\},$$

$$\therefore \frac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)}=\frac{x+(a+b+c)}{x-(a+b+c)},$$

$$\frac{x^3+px^2+qx+r}{x^3-px^2+qx-r}=\frac{x+p}{x-p}, \text{ where } \begin{cases} p=a+b+c, \\ q=ab+ac+bc, \\ r=abc. \end{cases}$$

$$\therefore \text{Art. } 195, \quad \frac{x^3+qx}{px^2+r}=\frac{x}{p},$$

$$px^3+pqx=px^3+rx,$$

$$pqx=rx, \quad \therefore x=0.$$

Also $pq=r$, or $(a+b+c)(ab+ac+bc)=abc$,

or $(a+b)(b+c)a+(a+b)bc+(b+c)ac+bc^2=abc$,

$$(a+b)(b+c)a+bc(b+c)+ac(b+c)=0,$$

$$(a+b)(b+c)a+(a+b)(b+c)c=0,$$

$$(a+b)(b+c)(a+c)=0,$$

$\therefore a+b=0$, or $b+c=0$, or $a+c=0$, any one of which will cause the proposed 'equation' to become an '*identity*'.

Ex. 8. If $(a^2+bc)^2(b^2+ac)^2(c^2+ab)^2=(a^2-bc)^2(b^2-ac)^2(c^2-ab)^2$, prove that either $a^3+b^3+c^3+abc=0$, or $a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1}=0$.

Multiply each of the equals by $a^2b^2c^2$, taking a into the 1st bracket, b into the 2nd, and c into the 3d; then

$$(a^2+abc)(b^2+abc)(c^2+abc)=\pm(a^2-abc)(b^2-abc)(c^2-abc),$$

or $\dfrac{(A+p)(B+p)}{(A-p)(B-p)}=\pm\dfrac{C-p}{C+p}$, writing A for a^2, B for b^2, C for c^2, and p for abc;

$$\text{or } \frac{AB+(A+B).p+p^2}{AB-(A+B).p+p^2}=\pm\frac{C-p}{C+p},$$

$$\text{Art. 195, } \frac{AB+p^2}{(A+B)p}=-\frac{C}{p}, \text{ or } -\frac{p}{C},$$

$$\therefore AB+AC+BC+p^2=0, \text{ or } (A+B+C)p^2+ABC=0,$$

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{p^2}{ABC}=0, \text{ or } A+B+C+\frac{ABC}{p^2}=0,$$

$$a^{-2}+b^{-2}+c^{-2}+a^{-1}b^{-1}c^{-1}=0, \text{ or } a^2+b^2+c^2+abc=0.$$

Ex. 9. Four bells commence tolling together, and toll at intervals of 18, 45, 81, 105 seconds respectively: what time will elapse before they toll again simultaneously?

Let x be the time required in seconds;

then, by the question, x is a multiple of each of the numbers 18, 45, 81, 105; $\therefore$ it is a *common* multiple of these numbers; and, by the question, it is their *Least* Common Multiple.

Now $18=2\times3\times3$, $45=3\times3\times5$, $81=3\times3\times3\times3$, $105=3\times5\times7$,

$\therefore$ L.C.M. required $=2\times3\times3\times3\times3\times5\times7$,

$$=\overset{\text{s.}}{5670}=\overset{\text{m.}}{94\tfrac{1}{2}}=\overset{\text{h.}}{1\tfrac{1}{2}}.\overset{\text{m.}}{4\tfrac{1}{2}}.$$

Ex. 10. A number of wheels commence rolling at the same moment from the same place in the same direction, travelling at the same rate, the number of feet in their circumferences being $a_1, a_2, a_3, \ldots a_n$, which are n different prime numbers. How far will they have travelled, when they have all first completed simultaneously numbers of entire revolutions, and how many revolutions will *each* have completed?

Since all travel over the same ground in the same time, let them all have *first* completed an integral number of revolutions, when each has travelled x feet; then

$\frac{x}{a_1}=$ a whole number $=$ number of revolutions of w_1,

$\frac{x}{a_2}=$ .... $=$ .... w_2,

....

$\frac{x}{a_n}=$ .... $=$ .... w_n,

$\therefore$ x is a common multiple of a_1, a_2, a_3, ... a_n, and the *least* common multiple. But since a_1, a_2, a_3, ... a_n are all prime numbers, their L.C.M. is the product of them all,

$$\therefore\ x=a_1a_2a_3\ldots a_n.$$

Also $\frac{x}{a_1}=a_2a_3\ldots a_n$, the number of revolutions by w_1; and similarly for the others.

Ex. 11. Find the sum of n such fractions as

$$\frac{1}{1+x},\ \frac{2x}{1+x^2},\ \frac{4x^3}{1+x^4},\ \frac{8x^7}{1+x^8},\ \&c.$$

$$\text{Since}\quad \frac{1}{1-x}-\frac{1}{1+x}=\frac{2x}{1-x^2},$$

$$\frac{2x}{1-x^2}-\frac{2x}{1+x^2}=\frac{4x^3}{1-x^4},$$

$$\frac{4x^3}{1-x^4}-\frac{4x^3}{1+x^4}=\frac{8x^7}{1-x^8},$$

$$\&c.\qquad \&c.\ =\ \&c.$$

$$\therefore\ \frac{1}{1-x}-\left(\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\&c.\ \text{to } n \text{ terms}\right)=\frac{mx^{m-1}}{1-x^m},$$

where $m=2^n$.

$$\therefore\ \frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\&c.\ \text{to } n \text{ terms}=\frac{1}{1-x}-\frac{mx^{m-1}}{1-x^m}.$$

Ex. 12. Find the product of n such binomials as

$$x+\frac{1}{x},\quad x^2+\frac{1}{x^2},\quad x^4+\frac{1}{x^4},\quad x^8+\frac{1}{x^8},\ \&c.$$

$$\text{Since} \quad (1-x)(1+x)=1-x^2,$$

$$(1-x^2)(1+x^2)=1-x^4,$$

$$(1-x^4)(1+x^4)=1-x^8,$$

$$(1-x^8)(1+x^8)=1-x^{16}, \text{ and so on,}$$

$\therefore\ (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8).\&c.\ldots(1+x^m)=1-x^{2m}$, where $m=2^n$,

$$\therefore\ (1+x^2)(1+x^4)(1+x^8).\&c. \text{ to } n \text{ factors}=\frac{1-x^{2m}}{1-x^2},$$

$$\therefore\ \left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right).\&c.\ldots\ldots\ldots\ldots=\frac{1-x^{2m}}{(1-x^2)x^{m-1}}.$$

Ex. 13. If a number contain n digits, prove that its square root contains $\frac{1}{2}n+\frac{1}{4}-\frac{1}{4}(-1)^n$ digits.

Since the square of a number of m digits contains either $2m$ or $2m-1$ digits, (Art. 371), conversely, the *square root* of a number containing $2m$ digits, or $2m-1$ digits, contains m digits; that is, the square root of a number containing n digits contains

$$\frac{n}{2} \text{ digits, when } n \text{ is even,}$$

$$\text{and } \frac{n-1}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{odd;}$$

both of which results are expressed by $\frac{1}{2}n+\frac{1}{4}-\frac{1}{4}(-1)^n$.

Ex. 14. On June 21, A.D. 1851 the Duke of Wellington had lived exactly 30,000 days. Find the day and year of his birth.

$$\frac{30000}{365}=82+\frac{70}{365},$$

$\therefore$ reckoning 365 days in a year, the Duke had lived 82 years and 70 days; and $\therefore$ he was born sometime in the year 1769.

Also, on the supposition of a leap-year occurring every 4th year, there would be 20 leap-years in 82 years; but 1800 was *not* a leap-year, therefore in these 82 years there were only 19 leap-years. Hence on June 21, 1769 the Duke was 70-19, or 51 days, old; that is, he was born on the 1st of May, 1769.

Ex. 15. Apply the method of proof called "demonstrative induction" to prove that

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2.$$

$$\text{Since}\quad 1^3+2^3=9=(1+2)^2,$$

$$\text{and}\quad 1^3+2^3+3^3=36=(1+2+3)^2,$$

assume that the law here indicated holds for a series of n terms, that is,

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2,$$

$$=\left\{\frac{n(n+1)}{2}\right\}^2,$$

then

$$1^3+2^3+3^3+\ldots n^3+(n+1)^3=\left\{\frac{n(n+1)}{2}\right\}^2+(n+1)^3,$$

$$=\frac{(n+1)^2}{4}.(n^2+4n+4),$$

$$=\left\{\frac{(n+1)(n+2)}{2}\right\}^2,$$

$$=\{1+2+3+\ldots+(n+1)\}^2.$$

Hence it is shewn that, if the law hold for a series of n terms, it holds also for a series of $(n+1)$ terms. But it has been shewn to hold for series of 2 terms and of 3 terms, therefore it holds for a series of 4 terms: and so on generally, or

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2, \text{ for all values of } n.$$

Ex. 16. If x be real, prove that $x^2-8x+22$ can never be less than 6.

$$\text{Let}\quad x^2-8x+22=m,$$

$$\text{then}\quad x^2-8x+16=m-6,$$

$$x-4=\pm\sqrt{m-6},$$

$$\therefore\ x=4\pm\sqrt{m-6};$$

in which $m-6$ must not be negative, that is, m must never be less than 6.

Ex. 17. Prove that the volume of a sphere, whose radius is 6 inches, is equal to the sum of the volumes of three other spheres whose radii are 3, 4, and 5 inches; given that the volume of a sphere varies as the cube of its radius.

Since the volume of a sphere $\propto$ (radius)$^3 = mr^3$, where m is the same for all spheres,

$$\therefore \text{ sum of volumes of the 3 spheres} = m.3^3 + m.4^3 + m.5^3,$$
$$= m.216,$$
$$= m.6^3,$$
$$= \text{volume of a sphere with radius 6.}$$

Ex. 18. Standard gold being coined at the rate of £3. 17*s.* $10\frac{1}{2}d.$ per oz., what is the least integral number of ounces that can be coined into an integral number of sovereigns?

Let x be the number of ounces of gold which will make an exact number (y) of sovereigns; then

$$£3.\ 17s.\ 10\tfrac{1}{2}d. \times x = 20s. \times y,$$
$$\text{or } 1869x = 480y,$$
$$\therefore\ x = \frac{480}{1869}.y = \frac{160}{623}.y,$$

the *least* integral value of which, $\because$ 160 and 623 are prime to each other, will be when $y = 623$; that is, 160 is the least integral value of x.

Ex. 19. Find m and n in terms of a and b, so that $\dfrac{ma+nb}{m+n}$ may be the Arith^c. Mean between m and n, and the Geom^c. Mean between a and b.

By the question, $\dfrac{ma+nb}{m+n} = \dfrac{m+n}{2} = \sqrt{ab}$,

$$\left.\begin{aligned} \therefore\ m+n = 2\sqrt{ab},\ \ mb+nb &= 2b\sqrt{ab}, \\ \text{and } ma+nb &= 2ab, \end{aligned}\right\}$$

$$m(a-b) = 2ab - 2b\sqrt{ab},$$

$$\therefore\ m = \frac{2b\sqrt{a}(\sqrt{a}-\sqrt{b})}{a-b} = \frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}.$$

$$\text{Similarly } n = \frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$$

Ex. 20. A vessel (A) contains a gallons of wine, and another (B) contains b gallons of water; c gallons are taken out of each and transferred to the other; and this operation is repeated any number of times. Shew that if $c=ab\div(a+b)$, the quantity of wine in each vessel will always remain the same after the first operation.

$$\text{Wine in } A \text{ after 1st operation} = a-c,$$

$$\text{......... } B \text{ ....} = c;$$

In 2nd operation, in taking c gallons of mixture from A, we take $\frac{a-c}{a}\times c$ gallons of wine; and in putting c gallons of mixture from B into A, we *add* $\frac{c}{b}\times c$ gallons of wine; so that the quantity of wine added will be the same as that subtracted, if

$$\frac{c}{b}=\frac{a-c}{a}, \text{ or } c=\frac{ab}{a+b}.$$

Ex. 21. In the last Problem shew that, if c be general in value, the quantity of wine in the second vessel after r operations will be

$$\frac{ab}{a+b}(1-p^r), \text{ where } c=\frac{ab}{a+b}(1-p).$$

$$\text{Wine in } B \text{ after 2nd operation} = c+\frac{a-c}{a}.c-\frac{c}{b}.c,$$

$$=c(1+p), \text{ if } p=1-c.\frac{a+b}{ab}, \text{ that is, if } c=\frac{ab}{a+b}.(1-p),$$

$$\therefore \text{ wine in } B \text{ after 2nd operation} = \frac{ab}{a+b}(1-p^2).$$

Assume the law suggested by the two values for the wine in B after the 1st and 2nd operations to hold for the $\overline{r-1}^{\text{th}}$ operation so that

$$\text{wine in } B \text{ after } \overline{r-1}^{\text{th}} \text{ operation} = \frac{ab}{a+b}(1-p^{r-1}),$$

$$\text{then wine in } A = a-\frac{ab}{a+b}(1-p^{r-1}),$$

$\therefore$ wine in B after r^{th} operation $= \frac{ab}{a+b}(1-p^{r-1})+\frac{c}{a}.\{a-\frac{ab}{a+b}(1-p^{r-1})\}$

$$-\frac{c}{b}\cdot\frac{ab}{a+b}(1-p^{r-1}),$$

$$=\frac{ab}{a+b}\{1-p^{r-1}+1-p-(1-p)(1-p^{r-1})\};$$

$$=\frac{ab}{a+b}(1-p^r).$$

Hence, if the law holds for $\overline{r-1}$ operations, it also holds for r: and therefore generally, since it was *shewn* to hold for two.

Ex. 22. Divide an odd number $2n+1$, into two other whole numbers, so that their product may be a *maximum*.

Let x be one part, then $2n+1-x=$ the other:

and, by the question, $(2n+1-x)x=m$, a maximum,

$$x^2-(2n+1)x=-m,$$

$$x^2-(2n+1)x+\left(\frac{2n+1}{2}\right)^2=\frac{4n^2+4n+1-4m}{4},$$

$$\therefore\ x=\frac{1}{2}\{2n+1\pm\sqrt{(4n^2+4n+1-4m)}\},$$

and the greatest value of m which will permit this to be *real* is when $4m=4n^2+4n$, in which case $x=\frac{1}{2}\{(2n+1)\pm1\}=n+1$, or n;

$\therefore$ the parts required are n, and $n+1$.

Ex. 23. If x be real, prove that $\frac{2x-7}{2x^2-2x-5}$ can have no value between $\frac{1}{11}$ and 1.

$$\text{Let } \frac{2x-7}{2x^2-2x-5}=m,$$

$$\text{then } 2x-7=2mx^2-2mx-5m,$$

$$2mx^2-(2m+2)x=5m-7,$$

$$x^2-\frac{m+1}{m}x+\left(\frac{m+1}{2m}\right)^2=\frac{1}{4m^2}(m^2+2m+1+10m^2-14m),$$

$$=\frac{1}{4m^2}(11m^2-12m+1),$$

$$=\frac{1}{4m^2}(11m-1)(m-1),$$

$$\therefore\ x=\frac{1}{2m}\{m+1\pm\sqrt{(11m-1)(m-1)}\},$$

Hence in order that x may be *real*, $11m-1$, and $m-1$, must be either both positive, or both negative, $\therefore$ m either $>\frac{1}{11}$ and >1, or <1 and $<\frac{1}{11}$, that is, can have no value between $\frac{1}{11}$ and 1.

Ex. 24. Eliminate a and b from the equations

$$\frac{a^3-x^3}{b^3-y^3}=\frac{2x+3y}{3x+2y},\quad a^3-b^3=(x-y)^3,\quad a^{\frac{2}{3}}+b^{\frac{2}{3}}=z^{\frac{2}{3}}.$$

$$\text{Here}\quad a^3-b^3=x^3-3x^2y+3xy^2-y^3,$$

$$\therefore\ a^3-x^3-(b^3-y^3)=3xy(y-x),\quad (\alpha),$$

$$1-\frac{b^3-y^3}{a^3-x^3}=\frac{3xy(y-x)}{a^3-x^3},$$

$$1-\frac{3x+2y}{2x+3y}=\frac{3xy(y-x)}{a^3-x^3},$$

$$\frac{1}{2x+3y}=\frac{3xy}{a^3-x^3},\quad \therefore\ a^3-x^3=6x^2y+9xy^2,$$

$$\therefore\ a^3=x^3+6xy^2+9x^2y,\qquad \therefore\ a^{\frac{2}{3}}=x^{\frac{2}{3}}+3x^{\frac{1}{3}}y.$$

Similarly, dividing (α) by b^3-y^3, $\qquad b^{\frac{2}{3}}=y^{\frac{2}{3}}+3xy^{\frac{1}{3}}.$

$$\therefore\ a^{\frac{2}{3}}+b^{\frac{2}{3}}=x^{\frac{2}{3}}+3xy^{\frac{1}{3}}+3x^{\frac{1}{3}}y+y^{\frac{2}{3}},$$

$$a^{\frac{2}{3}}+b^{\frac{2}{3}}=(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3=z^{\frac{2}{3}},$$

$$\therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=z^{\frac{2}{9}}.$$

Ex. 25. Eliminate m, n, p, q, from the equations

$$\frac{x-p}{m}+\frac{y+q}{n}=\frac{pm}{a^2}+\frac{qn}{b^2}=\frac{m^2}{a^2}-\frac{n^2}{b^2}=\frac{p^2}{a^2}+\frac{q^2}{b^2}-1=0.$$

Here $\frac{x}{m}+\frac{y}{n}=\frac{p}{m}-\frac{q}{n}$; but $\frac{pm}{a^2}=-\frac{qn}{b^2}$,

$$\text{and } \frac{m^2}{a^2}=\frac{n^2}{b^2},$$

$$\therefore \frac{p}{m}=-\frac{q}{n}.$$

$$\text{Also } \frac{m}{a}=\frac{n}{b},$$

$$\therefore \frac{x}{a}+\frac{y}{b}=\frac{2p}{a}=-\frac{2q}{b},$$

$$\text{and } 1=\frac{p^2}{a^2}+\frac{q^2}{b^2}=\frac{2p^2}{a^2}=\frac{1}{2}\left(\frac{x}{a}+\frac{y}{b}\right)^2,$$

$$\therefore \frac{x}{a}+\frac{y}{b}=\sqrt{2}.$$

Ex. 26. If the roots of the equation $ax^2+bx+c=0$ are in the ratio $m : n$, shew that $\frac{b^2}{ac}=\frac{(m+n)^2}{mn}$.

Let α, β, be the roots; then

$$\frac{\alpha}{\beta}=\frac{m}{n},\ \alpha+\beta=-\frac{b}{a},\ \text{and } \alpha\beta=\frac{c}{a},$$

$$\therefore \frac{\alpha+\beta}{\beta}=\frac{m+n}{n},\quad \therefore \beta=-\frac{n}{m+n}.\frac{b}{a},$$

$$\text{and } \alpha=\frac{m}{n}.\beta=-\frac{m}{m+n}.\frac{b}{a},$$

$$\therefore \alpha\beta=\frac{mn}{(m+n)^2}.\frac{b^2}{a^2}=\frac{c}{a},$$

$$\therefore \frac{b^2}{ac}=\frac{(m+n)^2}{mn}.$$

Ex. 27. Find the sum of all the numbers of the forms 121, 12321, 1234321, &c. in the scale whose radix is r.

$$121=r^2+2r+1=(r+1)^2,$$

$$12321=r^4+2r^3+3r^2+2r+1=r^2(r+1)^2+2r(r+1)+1,$$

$$=\{r(r+1)+1\}^2=(r^2+r+1)^2,$$

$$1234321=r^6+2r^5+3r^4+4r^3+3r^2+2r+1,$$

$$=r^4(r^2+2r+1)+2r^2(r^2+2r+1)+r^2+2r+1,$$

$$=(r^2+1)^2(r+1)^2=(r^3+r^2+r+1)^2;\ \text{and so on.}$$

$\therefore$ sum req^{d}. $=(r+1)^2+(r^2+r+1)^2+(r^3+r^2+r+1)^2+$&c. to 9 terms,

$$=\left(\frac{r^2-1}{r-1}\right)^2+\left(\frac{r^3-1}{r-1}\right)^2+\left(\frac{r^4-1}{r-1}\right)^2+\text{\&c.......}+\left(\frac{r^{10}-1}{r-1}\right)^2,$$

$$=\frac{1}{(r-1)^2}\{r^4+r^6\ldots+r^{20}-2r^2(1+r+r^2+\ldots r^{10})+9\},$$

$$=\frac{1}{(r-1)^2}\cdot\left\{\frac{r^4(r^{18}-1)}{r^2-1}-2r^2\cdot\frac{r^{11}-1}{r-1}+9\right\}.$$

Ex. 28. Shew that 12345654321 is divisible by 12321 in any scale of which the radix > 6.

$$12345654321=r^{10}+2r^9+3r^8+4r^7+5r^6+6r^5+5r^4+4r^3+3r^2+2r+1,$$

$$=(r^5+r^4+r^3+r^2+1)^2,\quad \text{(see last Ex.)}$$

$$=\left(\frac{r^6-1}{r-1}\right)^2.$$

Also $12321=r^4+2r^3+3r^2+2r+1$,

$$=(r^2+r+1)^2=\left(\frac{r^3-1}{r-1}\right)^2,$$

$$\therefore \frac{12345654321}{12321}=\left\{\frac{r^6-1}{r-1}\div\frac{r^3-1}{r-1}\right\}^2=(r^3+1)^2=\text{a whole number.}$$

Ex. 29. Find the series in A.P. of which the sum of the first n terms is equal to n^2, whatever be the value of n.

Let a be the 1st term, and b the com. diff.,

$$\text{then}\quad \{2a+(n-1)b\}.\frac{n}{2}=n^2,$$

$$2an+bn^2-bn=2n^2,$$

$$(b-2)n^2-(b-2a)n=0,\ \text{for all values of } n,$$

$$\left.\begin{aligned}\therefore\ b-2&=0, &\text{or } b&=2,\\ \text{and } b-2a&=0, &\text{or } a&=1,\end{aligned}\right\}\ \text{(Art. 340.)}$$

$\therefore$ the series is 1, 3, 5, 7, 9, &c.

Ex. 30. From the last Example deduce the integral solutions of the equation $x^2=y^2+z^2$.

By last Ex. $1+3+5+7+9$&c. to n terms $=n^2$, for all values of n.

Now in this series 9, 25, 49, 81, &c. are squares; let S_n stand for the sum of n terms.

1st. Then $S_5 = 5^2$,

$S_4 = 4^2$,

$S_5 = S_4 + 9$,

$\therefore\ 5^2 = 4^2 + 3^2, \quad \therefore\ x = 5, \quad y = 4, \quad z = 3.$

2nd. $S_{13} = 13^2$,

$S_{12} = 12^2$,

$S_{13} = S_{12} + 25$,

$\therefore\ 13^2 = 12^2 + 5^2, \quad \therefore\ x = 13, \quad y = 12, \quad z = 5:$

and so on.

Ex. 31. Shew that $\sqrt[m+n+p+q]{abcd}$ lies between the greatest and least of the quantities $\sqrt[m]{a}$, $\sqrt[n]{b}$, $\sqrt[p]{c}$, $\sqrt[q]{d}$.

Let $\sqrt[m+n+p+q]{abcd} = x$, then

$$\log x = \frac{\log a + \log b + \log c + \log d}{m+n+p+q},$$

which lies between the greatest and least of the quantities, (see Ex. 5, p. 134, *Alg.*)

$$\frac{\log a}{m}, \quad \frac{\log b}{n}, \quad \frac{\log c}{p}, \quad \frac{\log d}{q},$$

$$\text{or } \log \sqrt[m]{a}, \quad \log \sqrt[n]{b}, \quad \log \sqrt[p]{c}, \quad \log \sqrt[q]{d},$$

and $\because$ quantities increase or decrease as their logs. increase or decrease,

$\therefore\ x$ lies between the greatest and least of the quantities $\sqrt[m]{a}$, $\sqrt[n]{b}$, $\sqrt[p]{c}$, $\sqrt[q]{d}$.

Ex. 32. The sum of two numbers is 45, and their L.C.M. is 168; what are the numbers?

Let M be the G.C.M. of the numbers,

then $qM + q'M = 45$, q and q' being the quotients when divided by the G.C.M.,

and $qq'M =$ the L.C.M. $= 168$,

$$\therefore\ \frac{q+q'}{qq'} = \frac{45}{168} = \frac{15}{56},$$

from which we *see* at once that $q=8$, and $q'=7$;

$$\therefore\ M=\frac{45}{q+q'},=\frac{45}{15}=3,$$

and the numbers are 3×8, and 3×7, that is, 24 and 21.

Ex. 33. The G.C.M. of two numbers is 16, and their L.C.M. is 192; what are the numbers?

Let q, q' be the quotients of the two numbers divided by their G.C.M., then the numbers are $q\times 16$, $q'\times 16$; and

$$qq'\times 16=\text{the L.C.M.}=192,$$

$$\therefore\ qq'=\frac{192}{16}=12.$$

Now since q and q' are prime to each other, and 3×4 is the only way in which 12 can be produced from two prime factors,

$$\therefore\ q=3,\ \text{ and } q'=4,$$

$\therefore$ the required numbers are 3×16, and 4×16, or 48 and 64.

Ex. 34. Find x when $18x-3x^2>24$.

$$3(6x-x^2-8)>0,$$

$$3\{-(x^2-6x+8)\}>0,$$

$$3\{-(x-4)(x-2)\}>0,$$

$$\left.\begin{array}{l}\therefore\ x-4 \text{ must be negative, that is, } x<4,\\ \text{and } x-2 \text{ .... positive, ... } x>2,\end{array}\right\}\ \therefore\ x=3.$$

Ex. 35. Find all the integral values of x which satisfy the inequality $x^2<10x-16$,

$$-x^2+10x-16>0,$$

$$-(x^2-10x+16)>0,$$

$$-(x-2)(x-8)>0,$$

$$\therefore\ x>2, \text{ and } <8, \text{ that is } x=3,\ 4,\ 5,\ 6,\ 7.$$

Ex. 36. Find the greatest value of $\dfrac{x^2+14x+9}{x^2+2x+3}$.

$$\text{Let } \frac{x^2+14x+9}{x^2+2x+3}=m,$$

$$\text{then } (m-1)x^2+(2m-14)x=-3m+9,$$

$$x^2+\frac{2m-14}{m-1}\cdot x+\left(\frac{m-7}{m-1}\right)^2=\frac{-2m^2-2m+40}{(m-1)^2},$$

$$\therefore\ x=\frac{1}{m-1}\{7-m\pm\sqrt{-2m^2-2m+40}\},$$

$$=\frac{1}{m-1}\{7-m\pm\sqrt{-2(m+5)(m-4)}\},$$

in which m cannot exceed 4, but it may be equal to 4, $\therefore$ 4 is its *greatest* value.

Ex. 37. If x be real, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

$$\text{Let } \frac{x^2+34x-71}{x^2+2x-7}=m, \text{ then}$$

$$(m-1)x^2+2(m-17)x=7m-71,$$

$$\therefore\ x=\frac{1}{m-1}\{17-m\pm\sqrt{8(m-5)(m-9)}\};$$

hence, if x is to be real, $m-5$, and $m-9$, must both have the same sign, that is, either $m>9$, or <5, and therefore has no value between 5 and 9.

Ex. 38. Prove that $1.2.3\ldots n<\left(\dfrac{n+1}{2}\right)^n$.

1st. Let n be odd; then the middle factor of $1.2.3\ldots n$ is $\frac{1}{2}(n+1)$;

the p^{th} factor on the *left* of this $=\frac{1}{2}(n+1)-p$,

.... *right*......$=\frac{1}{2}(n+1)+p$,

and the product of these $=\left\{\frac{1}{2}(n+1)\right\}^2-p^2<\left\{\frac{1}{2}(n+1)\right\}^2$; and there will be $\frac{1}{2}(n-1)$ pairs of factors such as these; therefore

when n is odd, $1.2.3\ldots n<\left\{\frac{1}{2}(n+1)\right\}^{n-1}.\frac{1}{2}(n-1)$,

$$<\left\{\frac{1}{2}(n+1)\right\}^{n-1}.\frac{1}{2}(n+1),\ \textit{à fortiori},$$

$$<\left\{\frac{1}{2}(n+1)\right\}^{n}.$$

2nd. Let n be even; then the two middle factors of $1.2.3\ldots\ldots n$ will be $\frac{1}{2}n$ and $\frac{1}{2}(n+2)$, the product of which $=\frac{1}{4}n(n+2)$.

The factor p places to the *left* of the $\frac{1}{2}n^{\text{th}}$ is $\frac{1}{2}(n-p)$,

.... *right* .... $\frac{1}{2}(n+2)^{\text{th}}$ is $\frac{1}{2}(n+2)+p$,

and the product of these $=\frac{1}{4}n(n+2)-p-p^2<\frac{1}{4}n(n+2)$; and there are altogether $\frac{1}{2}n$ pairs of such factors; therefore

when n is even, $1.2.3\ldots n<\left\{\frac{1}{2^2}n(n+2)\right\}^{\frac{n}{2}}$,

$$<\left\{\frac{1}{2^2}(n+1)^2\right\}^{\frac{n}{2}},\ \textit{à fortiori},$$

$$<\left\{\frac{1}{2}(n+1)\right\}^{n},$$

$\therefore$ generally, $1.2.3\ldots n<\left\{\frac{1}{2}(n+1)^n\right\}$.

Ex. 39. Two smiths begin to strike their anvils together. The one (A) gives 12 strokes in 7 minutes; the other (B) 17 strokes in 9 minutes. What strokes of each most nearly coincide in the first half hour?

Suppose the x^{th} stroke of A actually coincident with the y^{th} stroke of B; then since x strokes of A occupy the same time as y strokes of B,

$$x\times\frac{12}{7}=y\times\frac{17}{9},$$

$$\text{or } \frac{x}{y}=\frac{119}{108}=1+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2}}}}$$

and the convergents are $\frac{1}{1}$, $\frac{10}{9}$, $\frac{11}{10}$, $\frac{54}{49}$, $\frac{119}{108}$, each of which fractions, proceeding from left to right, approaches nearer and nearer to $\frac{119}{108}$; and $\therefore$ $\frac{54}{49}$ is the nearest; but since A makes only 51 strokes in the half hour, $\frac{11}{10}$ is the fraction to be taken for the value of $\frac{x}{y}$, that is, the 11^{th} stroke of A and the 10^{th} of B are those which most nearly coincide within the given time.

Ex. 40. There are n points in a plane, no three of which are in the same straight line, with the exception of p which are all in the same straight line; find the number of triangles which result from joining them.

If no three points were in the same straight line, the number of triangles would be the number of combinations of n things taken 3 together, that is,

$$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}.$$

But since p points are all in the same straight line, which on the former supposition would among themselves make $\frac{p(p-1)(p-2)}{1\,.\,2\,.\,3}$ triangles; $\therefore$ the above number must be diminished by so much, or

$$\text{No. required}=\frac{n(n-1)(n-2)-p(p-1)(p-2)}{1\,.\,2\,.\,3}.$$

Ex. 41. Prove that $\frac{\lfloor n_1+n_2}{\lfloor n_1 . \lfloor n_2}$ is an integer; and deduce that $\frac{\lfloor n_1+n_2+n_3+\ldots+n_r}{\lfloor n_1 . \lfloor n_2 . \lfloor n_3 \ldots\ldots \lfloor n_r}$ is an integer.

By Art. 318, $\dfrac{\lfloor n}{\lfloor r \cdot \lfloor n-r} = \dfrac{n(n-1)\dots(n-r+1)}{1 \,.\, 2 \dots\dots r} =$ an Integer, by Art. 377, $\therefore$ writing n_1+n_2 for n, $\dfrac{\lfloor n_1+n_2}{\lfloor n_1 \cdot \lfloor n_2} =$ an Integer.

Again, $\dfrac{\lfloor n_1+n_2+n_3}{\lfloor n_1 \cdot \lfloor n_2 \cdot \lfloor n_3} = \dfrac{\lfloor n_1+\overline{n_2+n_3}}{\lfloor n_1 \cdot \lfloor n_2+n_3} \cdot \dfrac{\lfloor n_2+n_3}{\lfloor n_2 \cdot \lfloor n_3} =$ an Integer $\times$ an Integer,

$=$ an Integer;

and so on.

Ex. 42. Prove, without the aid of the Binomial Theorem, that if ${}_nC_r$ represent the number of combinations of n things r together,

$${}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n = 2^n - 1.$$

All the combinations of the n things $a_1, a_2, a_3 \dots a_n$, are contained in the coefficients of x, and of its powers, in the expansion of

$$(x+a_1)(x+a_2)(x+a_3)\dots(x+a_n).$$

And all the coefficients of x, and of its powers, excepting the coefficient of x^n, are exclusively combinations of the n things.

Hence the terms of the expansion of

$$(1+a_1)(1+a_2)(1+a_3)\dots(1+a_n),$$

excepting the number 1, are completely and exclusively combinations of the n things, $a_1, a_2, a_3, \dots a_n$.

Therefore the whole number of combinations is the value of $(1+a_1)(1+a_2)(1+a_3)\dots(1+a_n)-1$, when $a_1 = a_2 = a_3 = \&c. = a_n = 1$.

Hence, the whole number of combinations of n things,

$$\text{or } {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n = 2^n - 1.$$

Ex. 43. Along a straight line are placed n points. The distance between the first two points is 1 inch; and, generally, the distance between the r^{th} and $(r+1)^{\text{th}}$ points exceeds 1 inch by $\frac{1}{m}$th of the distance between the r^{th} and $(r-1)^{\text{th}}$ points. Find the distance between the last two points, and between the first and last

Ex. If $m=n=10$, shew that the distance between the extreme points is 9·87654321 inches.

The distance between points

(1) and (2), (2) and (3), (3) and (4) is

$$1\quad,\qquad 1+\frac{1}{m}\quad,\qquad 1+\frac{1}{m}+\frac{1}{m^2},\ \text{inches},$$

and generally between the r^{th} and $(r+1)^{\text{th}}$ points is

$$1+\frac{1}{m}+\frac{1}{m^2}+\ldots+\frac{1}{m^{r-1}},\ \text{or}\ \frac{1-m^{-r}}{1-m^{-1}},\ \text{inches}.$$

Hence the distance between the last two points, that is, the $(n-1)^{\text{th}}$ and the n^{th} is

$$\frac{1-m^{-(n-1)}}{1-m^{-1}}\ldots\ldots\ldots(1).$$

Again, the distance between the extreme points is

$$1+\frac{1-m^{-2}}{1-m^{-1}}+\frac{1-m^{-3}}{1-m^{-1}}+\ldots+\frac{1-m^{-(n-1)}}{1-m^{-1}}\ \text{inches},$$

$$=\frac{1}{1-m^{-1}}\{1-m^{-1}+1-m^{-2}+1-m^{-3}+\ldots+1-m^{-(n-1)}\}$$

$$=\frac{1}{1-m^{-1}}\left\{n-1-m^{-1}.\frac{1-m^{-(n-1)}}{1-m^{-1}}\right\}\ \text{inches}.$$

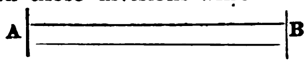
Ex. If $m=n=10$, the distance between the extreme points

$$=\frac{1}{1-10^{-1}}\{9-(10^{-1}+10^{-2}+\ldots+10^{-9})\},$$

$$=\frac{10}{9}\{9-0\cdot111111111\}=10-\frac{1}{9}\times(0\cdot111111111)=9\cdot87654321.$$

Ex. 44. Two straight rods, each c inches long, and divided into m and n equal parts respectively, where m and n are prime to one another, are placed in longitudinal contact, with their ends coincident. Prove that no two divisions are at a less distance than $\frac{c}{mn}$ inches; and that two pairs of divisions are at this distance.

Ex. If $m=250$, and $n=243$, find those divisions which are at the least distance.



Let $\left.\begin{matrix} d_1 \\ d_2 \end{matrix}\right\}$ be distances from A of the $\left\{\begin{matrix} x^{\text{th}} \text{ division of rod I,} \\ y^{\text{th}} \text{ .... II,} \end{matrix}\right.$

$$\text{then } d_1 = \frac{x}{m}.c, \quad d_2 = \frac{y}{n}.c,$$

$$\therefore\ d_1 - d_2 = c\left(\frac{x}{m} - \frac{y}{n}\right),$$

$$\text{or} \quad nx - my = \frac{mn}{c}(d_1 - d_2).$$

Now $nx-my$ is integral, and its least numerical value is unity, $\therefore$ the least value of $\frac{mn}{c}(d_1 \sim d_2)$ is 1, that is, the least value of $d_1 \sim d_2$ is $\frac{c}{mn}$.

Also there will be *two* pairs of such divisions, when

$$nx - my = 1, \text{ and when } nx - my = -1.$$

Ex. If $m=250$, and $n=243$, we have to solve the equations $243x - 250y = \pm 1$. From the former, applying the method of continued fractions, we get $x=107$, $y=104$, and from the latter, $x=143$, $y=139$.

Ex. 45. Two bells toll together for an hour; one bell tolls 244 times and the other 251 times, the first and last tolls of each taking place at the beginning and end of the hour respectively. Of the strokes (excluding the first and last) find those which are most nearly simultaneous: and determine a person's station in the straight line joining the bells, that those which are most nearly simultaneous of all may appear to him absolutely coincident; given that sound travels at the rate of 1080 feet per second.

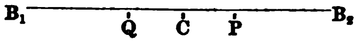
The 1st part of this Problem is precisely the Ex. in the last, since the tolls of the first bell divide the hour into 243 equal parts, and the second bell into 250 equal parts; in which case, as before shewn, those *divisions* most nearly coincident are

$$\left.\begin{matrix} \text{the } 104^{\text{th}} \text{ of I,} \\ \text{and the } 107^{\text{th}} \text{ of II,} \end{matrix}\right\} \qquad \left.\begin{matrix} \text{also the } 139^{\text{th}} \text{ of I,} \\ \text{and the } 143^{\text{rd}} \text{ of II,} \end{matrix}\right\}$$

$\therefore$ the tolls most nearly simultaneous are

$$\left.\begin{matrix} \text{the } 105^{\text{th}} \text{ of I,} \\ \text{and the } 108^{\text{th}} \text{ of II,} \end{matrix}\right\} \qquad \left.\begin{matrix} \text{also the } 140^{\text{th}} \text{ of I,} \\ 144^{\text{th}} \text{ of II.} \end{matrix}\right\}$$

Again, let B_1, B_2 be the bells: C the middle point between them: P the position in which a person must stand in order to hear the 105^{th} stroke of I and the 108^{th} of II simultaneously.



Then 105^{th} stroke of I happens $\frac{104}{243}\times 60\times 60$ seconds, and the 108^{th} stroke of II happens $\frac{107}{250}\times 60\times 60$ seconds, after the commencement of the hour.

Also sound travels to P from B_1 in $\frac{B_1P}{1080}$ seconds, and from B_2 in $\frac{B_2P}{1080}$ seconds.

$\therefore$ at P the 105^{th} stroke of I is heard $\frac{B_1P}{1080}+\frac{104}{243}\times 60\times 60$ seconds, and the 108^{th} stroke of II is heard $\frac{B_2P}{1080}+\frac{107}{250}\times 60\times 60$ seconds, after the commencement of the hour.

And these intervals must be the same,

$$\therefore\ 0=\frac{1}{1080}(B_1P-B_2P)+60\times 60\left(\frac{104}{243}-\frac{107}{250}\right),$$

$$\text{or}\quad 0=\frac{2CP}{1080}-\frac{60\times 60}{243\times 250},$$

$$\therefore\ CP=32\ \text{feet.}$$

Similarly if Q be the position in which a person must be in order to hear the 140^{th} stroke of I and the 144^{th} of II simultaneously,

$$CQ=32\ \text{feet.}$$

If the distance between the bells be less than 64 feet, the above investigation fails, and the points P, Q do not exist.

If the distance be 64 feet, the sound of the 105^{th} stroke of I will reach II just as it makes its 108^{th} stroke; hence the two strokes will be heard simultaneously by a person situated any where in the line B_1B_2 produced towards B_2. Similarly, the 140^{th} stroke of I and the 144^{th} of II will be heard simultaneously by a person situated anywhere in the line B_1B_2 produced towards B_1.

Ex. 46. Three bells commenced tolling simultaneously, and tolled at intervals of 25, 29, 33 seconds respectively. In less than half an hour the first ceased, and the second and third tolled 18 seconds and 21 seconds respectively after this, and then ceased. How many times did each bell toll?

Let $25t =$ N°. of seconds before bell I ceased,

$\therefore\ t+1 =$ N°. of times bell I tolled,

$$\therefore\ 1+\frac{25t+18}{29} = \ldots\ldots\ldots\ldots\ \text{II}\ \ldots\ldots = 1+x, \text{ suppose,}$$

$$\text{and } 1+\frac{25t+21}{33} = \ldots\ldots\ldots\ldots\ \text{III}\ \ldots\ldots = 1+y, \ \ldots\ldots$$

$$\therefore\ 29x-18 = 25t = 33y-21,$$

$$\text{or}\quad 33y-29x = 3, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore\ \left.\begin{aligned} x &= 33m-24,\\ y &= 29m-21,\end{aligned}\right\}\ \text{if } m=1,\ \left.\begin{aligned} x &= 9,\\ y &= 8,\end{aligned}\right\}\ \text{if } m=2,\ \left.\begin{aligned} x &= 42,\\ y &= 37.\end{aligned}\right\}$$

Again, $$29x-25t = 18, \ldots\ldots\ldots\ldots (2),$$

$$\therefore\ \left.\begin{aligned} x &= 25n-108,\\ t &= 29n-126,\end{aligned}\right\}\ \text{if } n=5,\ \left.\begin{aligned} x &= 17,\\ t &= 19,\end{aligned}\right\}\ \text{if } n=6,\ \left.\begin{aligned} x &= 42,\\ t &= 48.\end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned} t &= 48,\\ x &= 42,\\ y &= 37,\end{aligned}\right\}\ \text{satisfy all the conditions.}$$

Hence the first bell ceased 25×48 seconds, or 20 minutes, after it commenced tolling; and the bells tolled 49, 43, and 38 times respectively.

Ex. 47. Two particles, a and b, start together from the same point, and move in the same direction, along the same straight line; a moves with the uniform speed of 2 feet per second, whilst the speed of b is such that it moves over 1 foot the first second, and the number of feet described by it in any time *varies* as the square of the time. Prove that, during the motion, the particles will be 3 times at any given distance, less than a foot, from each other; and interpret the fourth result obtained in the algebraical solution of the problem.

Let s feet = distance between them at the end of t seconds,

then, as long as a is a-head of b,

$$s = 2t-t^2,\quad \text{or}\quad t = 1\pm\sqrt{1-s}.$$

Hence, as long as a is a-head of b, the greatest possible distance between them is 1 foot; and

$t_1 = 1 - \sqrt{1-s_1}$......(1) gives the *first* time they are at the given distance s_1. And after they have been at greatest distance,

$t_2 = 1 + \sqrt{1-s_1}$......(2) gives the *second* time.

Also they will be together again at the end of 2 seconds, (since $s=0$, when $t=2$), after which the equation is $s=t^2-2t$,

$$\text{or } t = 1 \pm \sqrt{1+s} \ldots\ldots (a),$$

$\therefore\ t_3 = 1 + \sqrt{1+s_1}$......(3) gives the *third* time of their being at the given distance s_1.

The *negative* sign in (a) makes t negative, and to get an interpretation of this write $-t$ for t in $s=t^2-2t$; then we have

$$s = t^2 + 2t,$$

which indicates, since s increases as t increases, that the answer in this case belongs to the question when the points are supposed to move in *opposite* directions.

Ex. 48. Supposing the receipts on a railway to vary as the increase of speed above 20 miles an hour, whilst the cost of working the trains varies as the square of that increase, and that at 40 miles an hour the expenses are just paid; find the speed at which the profits will be the greatest.

Let x be the speed at any time, in miles per hour,

then $x-20$ = its excess of 20 miles,

$$\therefore \text{ Receipts} \propto x-20 = m(x-20),$$

$$\text{Expenses} \propto (x-20)^2 = n(x-20)^2,$$

$$\text{when } x=40, \quad \text{Receipts} = 20m,$$

$$\text{Expenses} = 400n,$$

and, by the question, $20m = 400n$, $\therefore\ m = 20n$.

$$\text{And } m(x-20) - n(x-20)^2 = a \textit{ maximum},$$

$$\text{or } 20n(x-20) - n(x-20)^2 = \ldots\ldots\ldots$$

$$\text{or } 20(x-20) - (x-20)^2 = M, \text{ suppose},$$

$$(x-20)^2 - 20(x-20) + 100 = 100 - M,$$

$$\therefore\ x - 20 = 10 \pm \sqrt{100-M},$$

$$\therefore\ x = 30 \pm \sqrt{100-M},$$

and, in order that x may be real, M cannot be greater than, but may be equal to, 100, which is therefore its *greatest* value, in which case $x=30$.

Ex. 49. Determine whether the infinite series $\frac{a}{m+p}+\frac{a^2}{m+2p}+\frac{a^3}{m+3p}+\ldots$ is convergent or divergent,

$$\frac{(n+1)^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}}=a.\frac{m+np}{m+(n+1)p}=a\left\{1-\frac{p}{m+(n+1)p}\right\},$$

$=a$, when n is indefinitely increased. Hence the series will be convergent, if $a<1$, and divergent, if $a>1$.

Ex. 50. Prove that every quadratic surd, supposing the unit to be a line, may have an *exact geometrical* representation, although it cannot be exactly represented *arithmetically*.

Let ABC be an isosceles right-angled triangle, each of the equal sides AB, BC, being in length equal to the unit; then the hypothenuse AC represents $\sqrt{1+1}$, or $\sqrt{2}$. (Euclid, Book I. 49.)

Again, let $A'B'C'$ be a right-angled triangle, of which the side $A'B'=$ the unit, and $B'C'=AC$ in the former triangle,

$$\text{then } A'C'=\sqrt{1+2}=\sqrt{3}.$$

Again, let $A''B''C''$ be a right-angled triangle, of which $A''B''$ $=$ the unit, and $B''C''=2$ units, then

$$A''C''=\sqrt{1+4}=\sqrt{5}.$$

Again, let $A'''B'''C'''$ be a right-angled triangle, of which $A'''B'''=$ the unit, and $B'''C'''=A''C''$ in the former triangle, then

$$A'''C'''=\sqrt{1+5}=\sqrt{6};$$

and so on: by which process it is obvious that an exact *geometrical* representation can be constructed for any quadratic surd whatever.

Ex. 51. A person continually walks at an uniform speed, and always in the same direction, round the boundary of a square field; and another continually walks, at the same uniform speed, along a diagonal of the field, always retracing his steps when he reaches either extremity. Prove that, according to their relative initial positions, they will either (1) never meet at all, or (2) meet

once; but they will never meet more than once however long they continue to walk.

Call him who walks round the boundary β, and him who walks along the diagonal δ; and let β walk round in the direction $ABCD$, A, B, C, D being the angular points of the square in order of position, AC being the diagonal traversed by δ.

1st. They will meet at A, if δ begin walking *towards* A, and β at the same time be either at the *same* distance from A that δ is, or at the same distance $+ 2\times$diagonal: or, generally, if δ's distance from $A + 2m\times$diagonal $=\beta$'s dist. from $A + 4n\times$side of the square.

Again, they will meet at A, if δ begin walking towards C, and β be from A at a distance $=$ diagonal $+ \delta$'s dist. from C; or, generally, if δ's dist. from $C + \overline{2m+1}\times$diagonal $=\beta$'s dist. from $A + 4n\times$ side; m and n being positive integers.

Similarly they may meet at C.

But, if they once meet, since they describe equal spaces in equal times, and the side of a square is *incommensurable* with its diagonal, it is obvious that they can never meet again.

Also, if their positions at starting be any of those in which they are simultaneously, on the supposition that they *have* met once, then they will never meet at all.

Ex. 52. A sum of money in £. *s.* *d.* is multiplied by a certain number; the pence are now half what they were before, and the shillings and pounds each what the shillings were at first. What is the sum, and the multiplier?

Let the sum be £x. ys. zd. and m the unknown multiplier; then, by the question, $\frac{mz}{12} = p + \frac{\frac{1}{2}z}{12}$, where p is the number of shillings to be carried, and therefore a whole number,

$$\therefore \frac{(m-\frac{1}{2})z}{12} = p, \text{ a whole number,}$$

$$\text{or } (2m-1)z \text{ is divisible by } 24.$$

Now the factors of 24 are, 1×24, or 2×12, or 3×8, or 4×6,

$\left.\begin{array}{l} 2m-1=1, \\ \text{and } z=24, \end{array}\right\}$ is inadmissible, $\because$ $m=1$ is clearly impossible.

Also the 2nd, and 4th, pairs of factors are inadmissible, since

$2m-1$ cannot be even; $\therefore$ the only pair of factors are the 3rd, that is,

$$\left.\begin{array}{r}2m-1=3,\\ \text{and } z=8,\end{array}\right\} \qquad \left.\begin{array}{r}\therefore\ m=2,\\ \text{and } z=8.\end{array}\right\}$$

Hence the sum is £x. ys. 8d., which multiplied by 2 becomes £$2x$. $\overline{2y+1}$s. 4d.; but, by the question, $\dfrac{2y+1}{20}=q+\dfrac{y}{20}$,

$$\text{or } \frac{y+1}{20}=q, \text{ a whole N}^\circ\text{., to be carried to the £'s.}$$

And y cannot be greater than 19, $\therefore$ $y=19$, is the only admissible value of y, so that $y+1$ may be divisible by 20. We have then $2x+q=2x+1=19$, $\therefore$ $x=9$; and the sum required is £9. 19s. 8d., the multiplier being 2.

Ex. 53. If $\dfrac{N_1}{D_1}$, $\dfrac{N_2}{D_2}$ are any two consecutive convergents to $\dfrac{a}{b}$, shew that the error in taking $\dfrac{N_1}{D_1}$ for $\dfrac{a}{b} < \dfrac{1}{D_1D_2}$ but $> \dfrac{1}{D_1(D_1+D_2)}$.

Let $\dfrac{N_3}{D_3}$ be the next convergent after $\dfrac{N_2}{D_2}$, then $\dfrac{a}{b}$ differs from $\dfrac{N_3}{D_3}$ only by taking instead of the partial quotient q_3, the complete quotient $q_3+\dfrac{1}{\&c.}$, or Q, suppose; but

$$\frac{N_3}{D_3}=\frac{q_3N_2+N_1}{q_3D_2+D_1},\ \therefore\ \frac{a}{b}=\frac{QN_2+N_1}{QD_2+D_1},$$

$$\therefore\ \frac{a}{b}\sim\frac{N_1}{D_1}=\frac{QN_2+N_1}{QD_2+D_1}\sim\frac{N_1}{D_1}=\frac{Q(N_2D_1\sim N_1D_2)}{D_1(QD_2+D_1)},$$

$$=\frac{Q}{D_1(QD_2+D_1)}\ (\text{Art. } 349)=\frac{1}{D_1\left(D_2+\dfrac{D_1}{Q}\right)},$$

$$<\frac{1}{D_1D_2},\ \text{but} > \frac{1}{D_1(D_2+D_1)},\ \because\ Q>q_3, \text{ and } \therefore >1.$$

Ex. 54. If $\rho=\sqrt{\alpha^2+\beta^2}$ be defined to be the '*modulus*' of a binomial of the form $\alpha+\beta\sqrt{-1}$, where α, β, are rational; and ρ_1, ρ_2, ρ_3, ... ρ_n, be the moduli of n such binomials; prove that the modulus of the symbolical product of these n binomials is $\rho_1\rho_2\rho_3\ldots\rho_n$.

Let R_n represent the *modulus* of the product of n such binomials; then

$$(\alpha_1+\beta_1\sqrt{-1})(\alpha_2+\beta_2\sqrt{-1})=\alpha_1\alpha_2-\beta_1\beta_2+(\alpha_1\beta_2+\alpha_2\beta_1)\sqrt{-1},$$

$$\therefore\ (R_2)^2=(\alpha_1\alpha_2-\beta_1\beta_2)^2+(\alpha_1\beta_2+\alpha_2\beta_1)^2,$$

$$=(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2),$$

$$\therefore\ R_2=\rho_1\rho_2.$$

Similarly $R_3=R_2\rho_3$,

$R_4=R_2\rho_4$,

&c. = &c.

$R_n=R_{n-1}\rho_n$,

$\therefore$ by multip^n. $R_n=\rho_1\rho_2\rho_3\ldots\rho_n$.

Ex. 55. Explain the notation of "functions"; and shew that if $F(x)=a^x$, then $F(x).F(y)=F(x+y)$.

A *function* of x is any quantity containing x. It may contain other quantities besides x, but when considered only with reference to the manner in which x enters, and the value of x, it is called simply a function of x, and for shortness is written $F(x)$, or $f(x)$, or $\phi(x)$. Thus $ax+x^2$ is a function of x, and $ay+y^2$ is *the same* function of y, so that if $F(x)$ represent the former, $F(y)$ represents the latter.

In the Example given $F(x)=a^x$, $\therefore$ $F(y)=a^y$, and $\therefore$ $F(x).F(y)=a^{x+y}$ in which $\overline{x+y}$ enters in the same way that x does in a^x, that is,

$$F(x).F(y)=F(x+y).$$

Ex. 56. If $F(n, m)=\frac{1}{m}-n.\frac{1}{m+p}+\frac{n(n-1)}{1.2}.\frac{1}{m+2p}-\&c.$, shew that

$$F(n, m)=\frac{np}{m}.F(\overline{n-1}, \overline{m+p});$$

and thence deduce the sum of the series for $F(n, m)$, when n is a positive integer.

$$F(n, m)=\frac{1}{m}\{1-n.\frac{m}{m+p}+\frac{n(n-1)}{1.2}.\frac{m}{m+2p}-\&c.\},$$

$$=\frac{1}{m}\{1-n\left(1-\frac{p}{m+p}\right)+\frac{n(n-1)}{2}\left(1-\frac{2p}{m+2p}\right)-\&c.\}$$

$$=\frac{1}{m}\{1-n+\frac{n(n-1)}{1.2}-\&c.$$

$$+\frac{np}{m+p}-\frac{n(n-1)}{1.2}\cdot\frac{2p}{m+2p}+\&c.\},$$

$$=\frac{np}{m}\left\{\frac{1}{m+p}-(n-1)\cdot\frac{1}{m+2p}+\frac{(n-1)(n-2)}{1\ .\ 2}\cdot\frac{1}{m+3p}-\&c.\right\},$$

$$\{\because\ 1-n+\frac{n(n-1)}{1\ .\ 2}-\&c.=(1-1)^n=0,\}$$

$$=\frac{np}{m}\cdot F(\overline{n-1},\ \overline{m+p})\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, $F(\overline{n-1},\ \overline{m+p})=\dfrac{(n-1)p}{m+p}\cdot F(\overline{n-2},\ \overline{m+2p}),$

$$F(\overline{n-2},\ \overline{m+2p})=\frac{(n-2)p}{m+2p}\cdot F(\overline{n-3},\ \overline{m+3p}),$$

$$\&c.=\&c.$$

$$F\{\overline{n-(n-1)},\ \overline{m+\overline{n-1}\,.\,p}\}=\frac{p}{m+(n-1)p}\cdot F(\overline{n-n},\ \overline{m+np}),$$

$$F(0,\ \overline{m+np})=\frac{1}{m+np};\ n \text{ being a positive integer,}$$

$$\therefore\ F(n,\ m)=\frac{n(n-1)(n-2)\ldots\ldots 3.2.1}{m(m+p)(m+2p)\ldots(m+np)}\cdot p^n.$$

Ex. 57. The Julian Period consists of 7980 years; the cycles of the sun, moon, and the Indictions, of 28, 19, and 15 years, respectively. If any particular year be the n^{th} of the Julian period, the remainders, after the division of n by 28, 19, 15, are the years of the Solar and Lunar cycles, and of the Indictions, respectively. Prove the following Rule for determining the year of the Julian period when the years of the Solar and Lunar cycles and of the Indictions are given:

Multiply the number of the year in the Solar cycle by 4845, in the Lunar by 4200, and in the cycle of Indictions by 6916; divide the sum of these products by 7980, and the *remainder* is the year of the Julian period sought.

Let S be the year in the Solar cycle, L .... Lunar ... I .... Indictions, } corresponding to the n^{th} year of the Julian Period;

then
$$\left.\begin{aligned} n &= 28m+S, \ldots\ldots(1)\\ n &= 19p+L, \ldots\ldots(2)\\ n &= 15q+I, \ldots\ldots(3)\end{aligned}\right\}\text{ by the question.}$$

Multiply (1) by 4845, (2) by 4200, (3) by 6916; add together the resulting equations, divide by 7980, and reducing fractions, we have

$$2n+\frac{n}{7980}=17m+10p+13q+\frac{4845S+4200L+6916I}{7980},$$

and the fractions on each side must be equal,

$\therefore$ n = remainder after the division of $4845S+4200L+6916I$ by 7980.

Ex. 58. If an import duty of r shillings a quarter be laid on foreign corn when the price of corn in the English market is p shillings a quarter, and e, f, are the numbers of quarters of English and Foreign corn consumed in a year; and if the imposition of such a duty causes the price to rise to $p+\frac{r}{n}$ shillings a quarter, and the consumption to sink, so that the same sum of money is still expended, the English produce being supposed to remain constant; find the value of r most productive of revenue.

The total cost of the corn per annum when free of duty $=p(e+f)$,

$\therefore$ N°. of quarters *imported* after the duty is laid on $=\dfrac{p(e+f)}{p+\frac{r}{n}}-e,$

$$=\frac{npf-er}{np+r};$$

$$\therefore \text{ yearly revenue}=\frac{npf-er}{np+r}.r=a\ maximum.$$

Let x be its greatest value; then

$$npfr-er^2=npx+rx,$$

$$er^2-(npf-x)r=-npx,$$

$$\therefore\ r=\frac{1}{2e}\{npf-x\pm\sqrt{(npf-x)^2-4enpx}\}.$$

Now, in order that r may have a real value, $4enpx$ cannot be greater than, but may be equal to, $(npf-x)^2$, $\therefore$ the *greatest* value of x will be when

$$4enpx = (npf-x)^2, \text{ in which case } r = \frac{1}{2e}(npf-x).$$

$$\therefore\ 4e^2r^2 = 4enp\,(npf-2er),$$

$$4e^2r^2+4enp\,.\,2er+4e^2n^2p^2 = 4e^2n^2p^2\left(1+\frac{f}{e}\right),$$

$$2er = 2enp\,.\left\{\sqrt{1+\frac{f}{e}}-1\right\},$$

$$\therefore\ r = np\,.\left\{\sqrt{1+\frac{f}{e}}-1\right\}.$$

Ex. 59. A square sign-board is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these squares be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row?

W	*B*	*R*	*b*
B	*W*	*b*	*R*
R	*b*	*B*	*W*
b	*R*	*W*	*B*

Suppose *one* of the ways to be as in the annexed diagram. It is clear, that we cannot interchange the colours in any particular row, so as to satisfy the conditions; but we may permute *whole* rows, both vertical and horizontal, in all possible ways.

Now the number of ways in which the *vertical* rows can be permuted is $1\times2\times3\times4$. And the number of ways in which the *horizontal* rows can be permuted is also $1\times2\times3\times4$. And *each* of the former permutations may be combined with *each* of the latter; therefore, the whole number of the arrangements required is

$$\underline{|4}\times\underline{|4}, \text{ or } 576.$$

Ex. 60. Shew that the greatest coefficient, formed from the index n, in the expansion of $(a_1+a_2+a_3\ldots+a_m)^n$ is $\dfrac{\underline{|n}}{\left(\underline{|q}\right)^m.(q+1)^r}$, where q is the quotient and r the remainder, when n is divided by m.

Let q be the quotient, and r the remainder, when n is divided by m; and suppose $\alpha_1, \alpha_2, \ldots \alpha_m$ to be the exponents of a_1, $a_2, \ldots a_m$, in that term in the expansion of $(a_1+a_2+\ldots+a_m)^n$ which has the greatest coefficient. It follows that $\underline{|\alpha_1}.\underline{|\alpha_2}\ldots\underline{|\alpha_m}$ is not greater than $\underline{|\alpha_1-1}.\underline{|\alpha_2+1}\ldots\underline{|\alpha_m}$, and therefore that

$$\alpha_1 \text{ is not greater than } \alpha_2+1,$$

$$\text{not greater than } \alpha_3+1,$$

$$\ldots\ldots\ldots\ldots\ldots\alpha_m+1,$$

whence ma_1 is not greater than $a_1+a_2+\ldots a_m+m-1$,

i. e. not greater than $mq+r+m-1$,

$\therefore\ a_1$ is not greater than $q+1+\dfrac{r-1}{m}$,

and, similarly, is not less than $q-1+\dfrac{r+1}{m}$,

and therefore $=q$, or $q+1$;

and the same may be proved of the other exponents. Therefore $mq+r=n=a_1+a_2+\ldots a_m=mq+$ the number of indices that are equal to $q+1$; and therefore $r=$ the number of indices that are equal to $q+1$.

Therefore the greatest coefficient

$$=\frac{\lfloor n}{(\lfloor q)^{m-r}(\lfloor q+1)^r},$$

$$=\frac{\lfloor n}{(\lfloor q)^m(q+1)^r}.$$

MISCELLANEOUS EXAMPLES.

(THIRD SERIES.)

Ex. 1. $x^3-\sqrt[3]{6}.x=1.$

$$\text{Let } x=\frac{1}{\sqrt[3]{z}};$$

$$\therefore\ 1-\sqrt[3]{6z^2}=z;$$

$$\therefore\ 6z^2=1-3z+3z^2-z^3;$$

$$\therefore\ z^3+3z^2+3z+1=2;$$

$$\therefore\ z+1=\sqrt[3]{2};$$

$$\text{and } \therefore\ x=\sqrt[3]{\frac{1}{\sqrt[3]{2}-1}}.$$

Ex. 2. $x^3-3x=a^3+\frac{1}{a^3}$

$$=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right);$$

$$\therefore\ x^3-\left(a+\frac{1}{a}\right)^3=3\left\{x-\left(a+\frac{1}{a}\right)\right\};$$

$$\therefore\ x-\left(a+\frac{1}{a}\right)=0, \text{ or } x^2+\left(a+\frac{1}{a}\right)x+\left(a+\frac{1}{a}\right)^2=3;$$

$$\therefore\ x=a+\frac{1}{a}, \text{ or \&c.}$$

Ex. 3. $(x+a)(x+2a)(x+3a)(x+4a)=c^4.$

$$\text{Let } x+\frac{5a}{2}=y.$$

$$\therefore\left(y-\frac{3a}{2}\right)\left(y-\frac{a}{2}\right)\left(y+\frac{a}{2}\right)\left(y+\frac{3a}{2}\right)=c^4,$$

$$\left(y^2-\frac{9a^2}{4}\right)\left(y^2-\frac{a^2}{4}\right)=c^4,$$

$$y^4-\frac{5}{2}a^2y^2+\frac{9a^4}{16}=c^4,$$

$$y^2 = \frac{5}{4}.a^2 \pm \sqrt{c^4+a^4},$$

$$x = -\frac{5a}{2} \pm \sqrt{\frac{5a^2}{4} \pm \sqrt{c^4+a^4}}.$$

Ex. 4. $x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1=0.$

$$\therefore\ x^3+px^2+(p-1)x+\frac{1}{p-1}(x+p-1)=0;$$

$$\therefore\ (x+p-1)(x^2+x)+\frac{1}{p-1}(x+p-1)=0;$$

$$\therefore\ x+p-1=0, \text{ or } x^2+x+\frac{1}{p-1}=0;$$

$$\therefore\ x=1-p, \text{ or \&c.}$$

Ex. 5. $(p-1)^2x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1=0.$

$$\text{Let } x=\frac{1}{z};$$

$$\therefore\ (p-1)^2+pz+\left(p-1+\frac{1}{p-1}\right)z^2+z^3=0;$$

$$\therefore\ z^3+(p-1)z^2+\frac{1}{p-1}\{z^2+p(p-1)z+(p-1)^3\}=0;$$

$$\therefore\ z^2(z+p-1)+\frac{1}{p-1}(z+p-1)\{z+(p-1)^2\}=0;$$

$$\therefore\ z+p-1=0, \text{ or } z^2+\frac{1}{p-1}\{z+(p-1)^2\}=0;$$

$$\therefore\ x=\frac{1}{1-p}, \text{ or \&c.}$$

Ex. 6. $\left.\begin{aligned} x^2(y-z)&=a^3\\ y^2(z-x)&=b^3\\ z^2(x-y)&=c^3\end{aligned}\right\}.$

By addition,

$$\begin{aligned} a^3+b^3+c^3 &= x^2(y-z)+y^2(z-x)+z^2(x-y),\\ &= x^2y-xy^2-(x^2-y^2)z+(x-y)z^2,\\ &= \{xy-(x+y)z+z^2\}.(x-y),\\ &= -(x-y)(z-x)(y-z).\end{aligned}$$

But $a^3b^3c^3 = x^2y^2z^2(y-z)(z-x)(x-y)$;

$$\therefore\ x^2y^2z^2 = -\frac{a^3b^3c^3}{a^3+b^3+c^3} = R^2, \text{ suppose,}$$

$$\therefore\ xyz = R;$$

$$\begin{aligned}\therefore\ R+a^3 &= x\{yz+x(y-z)\},\\ R+b^3 &= y\{zx+y(z-x)\},\\ R+c^3 &= z\{xy+z(x-y)\},\\ R-a^3 &= x\{yz+x(z-y)\},\\ R-b^3 &= y\{zx+y(x-z)\},\\ R-c^3 &= z\{xy+z(y-x)\};\end{aligned}$$

$$\therefore\ \frac{R+a^3}{R-c^3} = \frac{x}{z}, \quad \frac{R+b^3}{R-a^3} = \frac{y}{x}, \quad \frac{R+c^3}{R-b^3} = \frac{z}{y};$$

$$\therefore\ \frac{x^2}{yz} = \frac{R^2-a^6}{(R+b^3)(R-c^3)};$$

$$\therefore\ x^3 = \frac{R^2-a^6}{(R+b^3)(R-c^3)}\cdot R.$$

Similarly, $y^3 = \dfrac{R^2-b^6}{(R+c^3)(R-a^3)}\cdot R$;

$$z^3 = \frac{R^2-c^6}{(R+a^3)(R-b^3)}\cdot R;$$

where $R^2 = -\dfrac{a^3b^3c^3}{a^3+b^3+c^3}$.

Ex. 7. $\left.\begin{aligned} x^2(y+z) &= a^3,\\ y^2(z+x) &= b^3,\\ xyz &= c^3.\end{aligned}\right\}$

By multiplication, $x^2y^2\{z^2+(x+y)z+xy\} = a^3b^3$;

by addition,
$$\left.\begin{aligned}\therefore\ x^2y^2(yz+zx+xy) &= a^3b^3-c^6,\\ x(yz+zx+xy) &= a^3+c^3,\\ y(yz+zx+xy) &= b^3+c^3;\end{aligned}\right\}$$

$$\left.\begin{aligned}\therefore\ xy^2 &= \frac{a^3b^3-c^6}{a^3+c^3},\\ x^2y &= \frac{a^3b^3-c^6}{b^3+c^3},\\ \frac{x}{y} &= \frac{a^3+c^3}{b^3+c^3};\end{aligned}\right\}$$

$$\therefore\ x^3=\frac{(a^3+c^3)(a^3b^3-c^6)}{(b^3+c^3)^2},$$

$$y^3=\frac{(b^3+c^3)(a^3b^3-c^6)}{(a^3+c^3)^2},$$

$$z^3=\frac{c^9}{x^3y^3}=\frac{c^9.(a^3+c^3)(b^3+c^3)}{(a^3b^3-c^6)^2}.$$

Ex. 8. $xyz=a^2(y+z)=b^2(z+x)=c^2(x+y)$;

$$\therefore\ \left.\begin{aligned}\frac{1}{xy}+\frac{1}{zx}&=\frac{1}{a^2},\\ \frac{1}{yz}+\frac{1}{xy}&=\frac{1}{b^2},\\ \frac{1}{zx}+\frac{1}{yz}&=\frac{1}{c^2};\end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned}\frac{2}{xy}&=\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2},\\ \frac{2}{zx}&=\frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2},\\ \frac{2}{yz}&=\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2};\end{aligned}\right\}$$

$\therefore$ dividing the 3rd by the product of the other 2, we have

$$x^2=2.\frac{\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}}{\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)};$$

and similarly y^2 and z^2.

Ex. 9. $xyz=a(y^2+z^2)=b(z^2+x^2)=c(x^2+y^2)$;

$$\therefore\ \left.\begin{aligned}\frac{y}{zx}+\frac{z}{xy}&=\frac{1}{a}\\ \frac{z}{xy}+\frac{x}{yz}&=\frac{1}{b}\\ \frac{x}{yz}+\frac{y}{zx}&=\frac{1}{c}\end{aligned}\right\};\qquad \therefore\ \left\{\begin{aligned}\frac{2x}{yz}&=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\\ \frac{2y}{zx}&=\frac{1}{c}+\frac{1}{a}-\frac{1}{b},\\ \frac{2z}{xy}&=\frac{1}{a}+\frac{1}{b}-\frac{1}{c};\end{aligned}\right.$$

$\therefore$ multiplying the last two together and inverting,

$$x^2=\frac{4}{\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)};$$

and similarly y^2 and z^2.

Ex. 10. $\left.\begin{array}{l}(x+y)(x+z)=a^2,\\(y+z)(y+x)=b^2,\\(z+x)(z+y)=c^2,\end{array}\right\}$

$$\left.\begin{array}{l}(y+z)^2=\dfrac{b^2c^2}{a^2},\\(z+x)^2=\dfrac{c^2a^2}{b^2},\\(x+y)^2=\dfrac{a^2b^2}{c^2};\end{array}\right\}$$

$$\therefore\ y+z=\pm\frac{bc}{a},$$

$$z+x=\pm\frac{ca}{b},$$

$$x+y=\pm\frac{ab}{c};$$

$$\therefore\ 2x=\pm\left(\frac{ab}{c}+\frac{ca}{b}-\frac{bc}{a}\right);$$

$$\therefore\ x=\pm\frac{abc}{2}\left(-\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

and similarly y and z.

Ex. 11. $\left.\begin{array}{l}x^3+y^3+z^3=3xyz\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\\3a-x+z=3b-y+x=3c-z+y\ldots\ldots(2)\end{array}\right\}.$

Each of the 3 expressions in (2) $=\frac{1}{3}$ of their sum,

$$=a+b+c;$$

$$\therefore\ \left.\begin{array}{l}z-x=b+c-2a\\x-y=a+c-2b\\y-z=a+b-2c\end{array}\right\}.$$

Let $x+y+z=u$;

$$\begin{aligned}\therefore\ u^3&=x^3+(y+z)^3+3x(y+z)(x+y+z),\\&=x^3+y^3+z^3+3yz(y+z)+3x(y+z)u,\\&=3xyz+3yz(y+z)+3x(y+z)u, \text{ by (1)},\\&=3yz.u+3x(y+z)u;\end{aligned}$$

$$\therefore\ u, \text{ or } x+y+z=0;$$

$$\text{also } x-y=a+c-2b,$$

$$y-z=a+b-2c;$$

$\therefore$ eliminating y and z by cross multiplication,

$$\text{we have } x=a-b;$$

$$\therefore y=b-c,$$

$$\text{and } z=c-a.$$

If we take the other values of u we have

$$u^2, \text{ or } (x+y+z)^2=3(xy+xz+yz);$$

$$\therefore x(x-y)+y(y-z)+z(z-x)=0;$$

$$\left.\begin{aligned} \therefore\ x(a+c-2b)+y(a+b-2c)+z(b+c-2a)&=0,\\ \text{and}\qquad x-y&=a+c+2b,\\ y-z&=a+b-2c,\end{aligned}\right\}$$

$$\therefore x.0=(b+c-2a)(a+b-2c)-(a+c-2b)^2;$$

$$\therefore x=\infty, \text{ (whence } y \text{ and } z \text{ also } =\infty),$$

$$\text{unless } \frac{b+c-2a}{a+c-2b}=\frac{a+c-2b}{a+b-2c},$$

$$\text{or } a(a-b)+b(b-c)+c(c-a)=0,$$

and then the values of x, y, z, are indeterminate.

Ex. 12. Obtain in the simplest form the value of x which satisfies the equation

$$(x+a-b-c-d)(x-a+b-c-d)(x-a-b+c-d)(x-a-b-c+d)$$
$$=(x-2a)(x-2b)(x-2c)(x-2d),$$

and investigate three equations independent of x, which include all those relations between a, b, c, d, for any one of which the equation becomes an identity.

$$\frac{(x+a-b-c-d)(x-a+b-c-d)}{(x-2a)(x-2b)}=\frac{(x-2c)(x-2d)}{(x-a-b+c-d)(x-a-b-c+d)}.$$

$$\therefore \text{each}=\frac{\text{difference of numerators}}{\text{difference of denominators}},$$

$$=\frac{(a-b-c-d)(-a+b-c-d)-4cd}{4ab-(-a-b+c-d)(-a-b-c+d)},$$

$$=\frac{(\overline{c+d}|^2-\overline{a-b}|^2)-4cd}{4ab-(\overline{a+b}|^2-\overline{c-d}|^2)}=\frac{\overline{c-d}|^2-\overline{a-b}|^2}{\overline{c-d}|^2-\overline{a-b}|^2},$$

$$=1, \text{ unless } c-d=a-b\ldots\ldots(\alpha),$$

$$-2(c+d)x+(a-b-c-d)(-a+b-c-d)=-2(a+b)x+4ab;$$

$$\therefore\ 2\{(a+b)-(c+d)\}x = 4ab-\overline{c+d}|^2+\overline{a-b}|^2,$$

$$= \overline{a+b}|^2-\overline{c+d}|^2;$$

$$\therefore\ x=\tfrac{1}{2}(a+b+c+d);$$

the exceptional case (a), which makes the equation evidently an identity, gives

$$a+d=b+c.$$

And so we may have $a+b=c+d$,

$$a+c=b+d;$$

these, being the only ways of combining the letters a, b, c, d, two and two, will be the three conditions, any one of which will make the equation an identity.

Ex. 13. Two numbers are as $3:5$, and their G.C.M. is 555: what are the numbers?

Since they are as $3:5$, they will also be so when divided by this G.C.M., in which case they become prime to each other. But 3 and 5 are prime to each other; ∴ the quotients after the division are 3 and 5; ∴ the numbers are 1665 and 2775.

Ex. 14. Each of two points moves uniformly in the circumference of a circle: one goes round 5 times while the other goes round twice; and at the end of 21590 days they return for the first time to the place from which they started. How long does each take to go round?

The first time they arrive together at their starting-point is when they have *actually* been round 5 and 2 times respectively; ∴ they take to go round $\frac{1}{5}$, and $\frac{1}{2}$, of 21590 days, i. e. 4318, and 10795, days.

Ex. 15. Four chronometers, which gain 6, 15, 27, 35, seconds a day respectively, shew true time. After how many days will this happen again?

If x be the number of days required, the total gain of the respective chronometers is $6x$, $15x$, $27x$, $35x$, seconds; and each of these must be a multiple of 12 hours, *i.e.* of $12\times60\times60^s$;

∴ x must be a multiple of $2\times60\times60$, $12\times4\times60$, $4\times20\times20$, $12\times12\times60$; the last being obtained from the consideration that as the factor 7 belonging to $35x$ does not appear in $12\times60\times60$, $5x$ must be a multiple of $12\times60\times60$.

The question evidently requires x to be as small as possible; ∴ x=the L.C.M. of $2\times60\times60$, $12\times4\times60$, $4\times20\times20$, $12\times12\times60$, *i.e.* 43200; ∴ they shew true time after the lapse of 43200 days.

Ex. 16. Two points revolve in the circumferences of two circles. At starting they are in a common diameter of the circles; and before arriving again simultaneously in their initial positions, they have been in common diameters n times, or m times, according as they have revolved in the same or in opposite directions. Prove that $m+n$ and $m-n$ must have a common divisor 4, and no higher common divisor; and that the ratio of the periods of revolution of the points is $m+n : m-n$.

The problem will be the same as if the circles were coincident, which we shall therefore suppose. Let $2c$ be the circumference, a and b the times which they respectively take for a revolution. Then in a unit of time they describe $\frac{2c}{a}$, $\frac{2c}{b}$, and are therefore separated by an interval $\frac{2c}{b}-\frac{2c}{a}$, over and above the distance already between them; therefore the time that elapses between two successive instants when they are in common diameters, is

$$\frac{c}{\frac{2c}{b}-\frac{2c}{a}}=\frac{ab}{2(a-b)}.$$

And in this time they describe

$$\frac{b}{a-b}c, \quad \frac{a}{a-b}c;$$

therefore, by question,

$$\frac{nb}{a-b}c, \quad \frac{na}{a-b}c$$

are multiples of $2c$, $=k.2c$, $k'.2c$, suppose.

So, if they moved in opposite directions,

$$\frac{mb}{a+b}c, \quad \frac{ma}{a+b}c$$

are multiples of $2c$, $=k.2c$, $k'.2c$;

for as far as regards their arriving simultaneously at their initial positions, the directions of their motions are immaterial.

$$\left.\begin{aligned}\therefore\ \frac{nb}{a-b}&=2k=\frac{mb}{a+b},\\ \frac{na}{a-b}&=2k'=\frac{ma}{a+b},\end{aligned}\right\}$$

where k, k' are prime to each other, by the conditions of the question;

$$\therefore\ n = 2(k'-k), \quad m = 2(k'+k);$$

$$\therefore\ m+n = 4k', \quad m-n = 4k,$$

whence $m+n$, $m-n$ have 4 for their G.C.M., and the ratio of the periods

$$= a : b = k' : k = m+n : m-n.$$

Ex. 17. Three particles start at the same point to move uniformly in the circumference of a circle, and the period of the first : that of the second : that of the third :: a prime number : a second prime number : a third prime number. The first and second arrive together at the starting-point 6 min. before the first and third, and 28 min. before the second and third arrive there together; also when the first and second arrive there together, they have been together as many times as the second and third when they arrive there together; and all three arrive there together 1 h. 55 min. 30 sec. after starting. Determine the periods of revolution.

Let ka, kb, kc, be the periods expressed in minutes; a, b, c, being prime numbers: and let s be the circumference.

By a method similar to that in preceding question, time between two successive instants when (1) and (2) are together $= \dfrac{kab}{b-a}$; in which time (1) describes $\dfrac{b}{b-a}s$, and (2) describes $\dfrac{a}{b-a}s$. When they arrive together at the starting-point, (1) will evidently have described bs, and (2) as, which will be in time kab, and they have then been together $b-a$ times.

So (1) and (3) will arrive together at starting-point at the end of time kac, having been together $c-a$ times.

And (2) and (3) will arrive there together in time kbc, having been together $c-b$ times.

They all arrive at starting-point together in time $kabc$;

$$\therefore \text{ by question, } kab = kac-6, \quad \text{or } ka(c-b) = 6,$$

$$kab = kbc - 28, \quad \text{or } kb(c-a) = 28,$$

$$kabc = 115\tfrac{1}{2},$$

$$b-a = c-b;$$

$$\therefore \left.\begin{aligned} \frac{c-b}{bc} &= \frac{6}{115\frac{1}{2}} \\ \frac{c-a}{ca} &= \frac{28}{115\frac{1}{2}} \end{aligned}\right\}, \text{ or } \left\{\begin{aligned} \frac{1}{b} - \frac{1}{c} &= \frac{4}{77}, \\ \frac{1}{a} - \frac{1}{c} &= \frac{8}{33}; \end{aligned}\right.$$

$$\therefore \text{ by subtraction, } \frac{1}{a}-\frac{1}{b}=\frac{4}{21};$$

$$\therefore \frac{c-b}{bc} \div \frac{b-a}{ab} = \frac{4}{77} \div \frac{4}{21},$$

$$\text{or } \frac{a}{c}=\frac{3}{11}.$$

But a and c are prime; $\therefore a=3$, $c=11$,

and $\therefore b=7$;

$\therefore$ also $k=\frac{1}{2}$,

and the periods are $1\frac{1}{2}$, $3\frac{1}{2}$, $5\frac{1}{2}$, minutes.

Ex. 18. If the G.C.M. of $m+n$ and $m-n$ be 4: prove that the G.C.M. of m and n will be either 2 or 4; and that when it is 2, each of $\frac{m}{2}$, $\frac{n}{2}$, is odd; and when it is 4, one of $\frac{m}{4}$, $\frac{n}{4}$, is odd, and the other even.

The G.C.M. of $m+n$ and $m-n$, viz. 4, measures their sum and difference, *i. e.* $2m$ and $2n$; $\therefore$ m and n must be divisible by 2.

Also the G.C.M. of m and n measures $m+n$ and $m-n$, and therefore measures 4; therefore G.C.M. of m and n is either 2 or 4.

(1) Let it be 2; then one of $\frac{m}{2}$, $\frac{n}{2}$, is odd (for no two even numbers are prime to each other); and if the other is even, $\frac{m+n}{2}$, $\frac{m-n}{2}$, are both odd, which is impossible; therefore $\frac{m}{2}$, and $\frac{n}{2}$, are both odd.

(2) Let it be 4; then one of $\frac{m}{4}$, $\frac{n}{4}$, is odd; and if the other be odd, $\frac{m+n}{4}$, $\frac{m-n}{4}$, are both even, which is impossible; therefore one of $\frac{m}{4}$, $\frac{n}{4}$, is odd, and the other even.

Ex. 19. From the longer of two rods a piece equal to the shorter is cut off: with the shorter and the remainder of the longer a similar operation is performed; and so on, until the rods are of equal length. Prove that if this last length be taken for the unit of measurement, the lengths of the original rods will be represented by numbers prime to each other. Also,

Explain that there are many relative lengths of the original rods for which the cut rods will never become equal, however long the operation be continued.

As division is shortened subtraction, it is evident that the operation in question is the same as that of dividing the longer by the shorter, and the shorter by the remainder, and so on, until there is no remainder: the last length spoken of will be the last divisor. This is the same operation as that of finding the G.C.M. of two numbers representing the number of some common unit contained in the rods. If the unit of measurement spoken of in the question be taken, the numbers which represent the original rods will be those which express the number of times this G.C.M. is contained in them, which of course are prime to each other.

It is not difficult to see that in many cases this operation will never terminate, as for example, if the rods are equal to the diagonal and side of a square, the diagonal of a cube and that of its face, &c.

Ex. 20. Define commensurable magnitudes; and prove that magnitudes which are commensurable with the same magnitude are commensurable with one another.

Commensurable magnitudes are those which can be obtained by the repetition of a certain the same magnitude, or which have a common measure.

Let B and C be both commensurable with A, *i. e.* let $A=mk$, $B=nk$, where m and n are integers: and $A=m'k'$, $C=n'k'$, where m' and n' are integers. Also let $k=m'\lambda$; $\therefore m'k'=A=mk=mm'\lambda$, or $k'=m\lambda$, whence $B=nm'\lambda$, $C=n'm\lambda$, *i. e.* B and C are both multiples of some quantity λ, and therefore are commensurable with each other.

Ex. 21. The adjacent sides of a rectangular parallelogram are respectively equal to the hypothenuses of two right-angled triangles whose sides are commensurable with the unit of linear measurement. Prove that its area will be commensurable or incommensurable with the corresponding unit of square measurement, according as its sides are or are not commensurable with each other.

The squares of the sides of the parallelogram will be equal to the sum of the squares of the sides of the triangles containing the right angles; and therefore the sides of the parallelogram will be expressed by the square roots of two numbers. As the area is represented by the product of the sides, it will be commensurable with the unit of square measurement, if this product involves no surd factor, *i.e.* if the expressions for the sides are

either rational, or can be reduced so as to have the same surd factor, (for the product of two dissimilar surds cannot be rational). In the first case both the sides are commensurable with the unit of linear measurement; in the second case, they can be both obtained by the repetition of the line represented by the surd factor; therefore in both cases they are commensurable with each other. Q.E.D.

Ex. 22. Shew how to find *geometrically* lines equal to the G.C.M. and the L.C.M. of two given commensurable straight lines.

For finding the G.C.M., see Ex. 19, above.

For the L.C.M. If the lines a and $b=mk$, nk, where k is their G.C.M.; and therefore m and n are numbers prime to each other, the L.C.M. of the two is mnk, i.e. na, or mb: whence we must find the G.C.M. of the two lines, and repeat the one as often as this G.C.M. is contained in the other. See Euclid, Book VII. Prop. 2 and 36, and Book X. Prop. 3.

Ex. 23. Prove that

$$\frac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=(a+b)(b+c)(c+a).$$

The denominator

$$=a^2(b-c)+bc(b-c)-a(b^2-c^2),$$
$$=(b-c).\{a^2-a(b-c)+bc\},$$
$$=(b-c)(a-b)(a-c).$$

In like manner the numerator

$$=(b^2-c^2)(a^2-b^2)(a^2-c^2);$$

therefore the fraction

$$=(a+b)(b+c)(a+c)^*.$$

Ex. 24. Obtain the continued product of

$$a+b+c+d,\quad a+b-c-d,\quad a-b-c+d,\quad a-b+c-d;$$

also of

$$-a+b+c+d,\quad a-b+c+d,\quad a+b-c+d,\quad a+b+c-d:$$

and shew that the sum of these products $=16abcd$.

* This might have been solved thus:—If c were put $=b$, the denominator $=0$, and therefore it is divisible by $b-c$; so also by $a-b$, and by $c-a$; these being prime to each other, it must be divisible by $(a-b)(b-c)(c-a)$, and therefore $=k(a-b)(b-c)(c-a)$, where k cannot contain a, b or c, owing to the dimensions of the denominator and this divisor. By comparing the numerator with the denominator we shall have the numerator $=k(a^2-b^2)(b^2-c^2)(c^2-a^2)$: whence the required result is obtained. This will also apply to Ex. 27, below.

The first product

$$=(a^2+2ab+b^2-c^2-2cd-d^2)(a^2-2ab+b^2-c^2+2cd-d^2),$$

$$=(a^2+b^2-c^2-d^2)^2-4(ab-cd)^2,$$

$$=a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2-2b^2c^2-2b^2d^2-2c^2d^2+8abcd.$$

The second is obtained from the first by writing $-a$ for a, and changing the sign of the whole; and therefore it

$$=-a^4-b^4-c^4-d^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2+2c^2d^2+8abcd,$$

and the sum of the two evidently $=16abcd$.

Ex. 25. Find the continued product of n such trinomials as

x^2-ax+a^2, $x^4-a^2x^2+a^4$, $x^8-a^4x^4+a^8$, $x^{16}-a^8x^8+a^{16}$, &c.

The respective trinomials

$$=\frac{x^4+a^2x^2+a^4}{x^2+ax+a^2},\quad \frac{x^8+a^4x^4+a^8}{x^4+a^2x^2+a^4},\quad \frac{x^{16}+a^8x^8+a^{16}}{x^8+a^4x^4+a^8},\ \&c.:$$

the last of them is $x^m-a^{\frac{m}{2}}x^{\frac{m}{2}}+a^m$,

where $m=2^n$, and $=\dfrac{x^{2m}+a^mx^m+a^{2m}}{x^m+a^{\frac{m}{2}}x^{\frac{m}{2}}+a^m}$.

The product of these is evidently

$$\frac{x^{2m}+a^mx^m+a^{2m}}{x^2+ax+a^2}.$$

Ex. 26. Find the sum of n such fractions as

$$\frac{2x-a}{x^2-ax+a^2},\quad \frac{4x^3-2a^2x}{x^4-a^2x^2+a^4},\quad \frac{8x^7-4a^4x^3}{x^8-a^4x^4+a^8},\ \&c.$$

We have $\dfrac{2x+a}{x^2+ax+a^2}+\dfrac{2x-a}{x^2-ax+a^2}=\dfrac{2x(2x^2+a^2)}{x^4+a^2x^2+a^4}$,

so $$\frac{2x(2x^2+a^2)}{x^4+a^2x^2+a^4}+\frac{2x(2x^2-a^2)}{x^4-a^2x^2-a^4}=\frac{4x^3(2x^4+a^4)}{x^8+a^4x^4+a^8},$$

$$\frac{4x^3(2x^4+a^4)}{x^8-a^4x^4+a^8}+\frac{4x^3(2x^4-a^4)}{x^8-a^4x^4+a^8}=\frac{8x^7(2x^8+a^8)}{x^{16}+a^8x^8+a^{16}},$$

the second of the fractions in the left-hand members being those in the given series. We shall at last have

$$\frac{2^{n-1}.x^{\frac{m}{2}-1}\left(2x^{\frac{m}{2}}+a^{\frac{m}{2}}\right)}{x^m+a^{\frac{m}{2}}x^{\frac{m}{2}}+a^m}+\frac{2^{n-1}.x^{\frac{m}{2}-1}\left(2x^{\frac{m}{2}}-a^{\frac{m}{2}}\right)}{x^m-a^{\frac{m}{2}}x^{\frac{m}{2}}+a^m}=\frac{2^n.x^{m-1}(2x^m+a^m)}{x^{2m}+a^mx^m+a^{2m}},$$

where $m=2^n$.

Adding these all up, and rejecting those that are common to both sides of the equality, we have

$$\frac{2x+a}{x^2+ax+a^2}+\text{required sum}=\frac{2^n.x^{m-1}(2x^m+a^m)}{x^{2m}+a^mx^m+a^{2m}};$$

$$\therefore\ \text{required sum}=\frac{2mx^{2m-1}+ma^mx^{m-1}}{x^{2m}+a^mx^m+a^{2m}}-\frac{2x+a}{x^2+ax+a^2}.$$

Ex. 27. Prove that

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a).$$

The denominator

$$=-3\{ab(a-b)+bc(b-c)+ca(c-a)\},$$
$$=-3\{a^2(b-c)-a(b^2-c^2)+bc(b-c)\},$$
$$=-3(b-c)\{a^2-(b+c)a+bc\},$$
$$=-3(b-c)(a-b)(a-c).$$

So the numerator

$$=-3(b^2-c^2)(a^2-b^2)(a^2-c^2);$$

therefore the fraction

$$=(a+b)(b+c)(c+a).$$

Ex. 28. For what unit of time will the durations 11574[s]. and 360360[s]. be represented by numbers prime to each other? Resolve the same question for 3·6 hours and 2·76 hours.

The unit required must evidently be that magnitude which is represented by the G.C.M. of the numbers representing the proposed quantities. Since quantities cannot be compared unless they are expressed in terms of the same unit, it will be necessary to have them so expressed before the question can be solved.

For (1), the unit will be that number of seconds expressed by the G.C.M. of 11574 and 360360, *i.e.* 18 seconds.

For (2), if the quantities be referred to the hundredth part of an hour as an unit, they are represented by 360 and 276. The G.C.M. of these is 12, therefore the unit of time required is 12 hundredth parts of an hour, *i.e.* 432 seconds

Ex. 29. Divide $1+2x^{2n+1}+x^{4n+2}$ by $1+2x+x^2$: writing down the $(2n-m+1)^{\text{th}}$ and $(2n+m+1)^{\text{th}}$ terms in the quotient; the $(2n-m)^{\text{th}}$ and $(2n+m)^{\text{th}}$ remainders; and the complete quotient.

$$
\begin{array}{l}
1+2x+x^2)\,1+2x^{2n+1}+x^{4n+2}\,(1-2x+3x^2-\&c. \\
\qquad\qquad 1+2x+x^2 \\
\hline
\qquad\qquad -2x-x^2 \\
\qquad\qquad -2x-4x^2-2x^3 \\
\hline
\qquad\qquad\qquad 3x^2+2x^3 \\
\qquad\qquad\qquad 3x^2+6x^3+3x^4 \\
\hline
\qquad\qquad\qquad\quad -4x^3-3x^4 \\
\qquad\qquad\qquad\quad \&c.
\end{array}
$$

The law to which the remainders and terms in the quotient are subject will hold good until any fresh terms are taken in from the dividend.

The remainders are alternately negative and positive; therefore the $(2n-m)^{\text{th}}$ has the sign $(-1)^{2n-m}$, *i.e.* $(-1)^m$.

The coefficient of the second term is the number of the remainder, and less by 1 than the coefficient of the first term, and so on;

$\therefore$ $(2n-m)^{\text{th}}$ remainder is $(-1)^m\{(2n-m+1)x^{2n-m}+(2n-m)x^{2n-m+1}\}$, and the two terms of the dividend, $+2x^{2n+1}+x^{4n+2}$, must be added in.

The $(2n-m+1)^{\text{th}}$ term in the quotient will be

$$(-1)^m(2n-m+1)x^{2n-m}.$$

Proceeding in this manner, we have

$$
\begin{array}{ll}
(2n-1)^{\text{th}}\text{ rem}^{\text{r}}\ldots\ldots & -2nx^{2n-1}-(2n-1)x^{2n}+2x^{2n+1}+x^{4n+2} \\
 & -2nx^{2n-1}-4nx^{2n}-2nx^{2n+1} \\
\hline
(2n)^{\text{th}}\text{ rem}^{\text{r}}\ldots\ldots & (2n+1)\,x^{2n}+(2n+2)x^{2n+1}+x^{4n+2} \\
 & (2n+1)x^{2n}+(4n+2)\,x^{2n+1}+(2n+1)x^{2n+2} \\
\hline
(2n+1)^{\text{th}}\text{ rem}^{\text{r}}\ldots\ldots & -2nx^{2n+1}-(2n+1)\,x^{2n+2}+x^{4n+2} \\
 & -2n^{2n+1}-4nx^{2n+2}-2nx^{2n+3} \\
\hline
(2n+2)^{\text{th}}\text{ rem}^{\text{r}}\ldots\ldots & (2n-1)x^{2n+2}+2nx^{2n+3}+x^{4n+2} \\
 & \&c.;
\end{array}
$$

$$
\begin{array}{ll}
(2n)^{\text{th}}\text{ term in quo}^{\text{t}}\ldots\ldots & -2nx^{2n-1}, \\
(2n+1)^{\text{th}}\text{ term}\ldots\ldots\ldots\ldots\ldots & +(2n+1)x^{2n}, \\
(2n+2)^{\text{th}}\text{ term}\ldots\ldots\ldots\ldots\ldots & -2nx^{2n+1}, \\
\&c.
\end{array}
$$

$\therefore$ $(2n+m)^{\text{th}}$ remr. is $(-1)^m\{(2n-m+1)x^{2n+m}+(2n-m+2)x^{2n+m+1}\}+x^{4n+2}$;

$\therefore$ $(2n+m+1)^{\text{th}}$ term in quotient is $(-1)^m(2n-m+1)x^{2n+m}$.

There will be $4n+1$ terms in the quotient; and the $(4n)^{\text{th}}$ remainder will be found by putting $m=2n$, and therefore is

$$+x^{4n}+2x^{4n+1}+x^{4n+2},$$

whence the last term in the quotient is $+x^{4n}$, and final remainder 0.

Ex. 30. Eliminate x, y, z from the equations

$$\left.\begin{aligned}(x-y)(y-z)(z-x)&=a^3\ldots.(1),\\(x+y)(y+z)(z+x)&=b^3\ldots.(2),\\(x^2+y^2)(y^2+z^2)(z^2+x^2)&=c^6\ldots.(3),\\(x^4+y^4)(y^4+z^4)(z^4+x^4)&=p^{12}\ldots(4).\end{aligned}\right\}$$

From (1), $\dfrac{a^3}{xyz}=\left(1-\dfrac{y}{x}\right)\left(1-\dfrac{z}{y}\right)\left(1-\dfrac{x}{z}\right)$,

$$=-\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)+\left(\frac{z}{y}+\frac{y}{z}+\frac{z}{x}\right).$$

From (2), $\dfrac{b^3}{xyz}=\left(1+\dfrac{y}{x}\right)\left(1+\dfrac{z}{y}\right)\left(1+\dfrac{x}{z}\right)$,

$$=2+\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)+\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ldots\ldots\ldots(a);$$

$$\therefore\ \frac{a^3+b^3}{2xyz}=1+\frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

From (1), (2), and (3),

$$\left.\begin{aligned}(x^2-y^2)(y^2-z^2)(z^2-x^2)&=a^3b^3,\\(x^2+y^2)(y^2+z^2)(z^2+x^2)&=c^6,\end{aligned}\right\}$$

which are analogous to (1) and (2);

$$\therefore\ \frac{a^3b^3+c^6}{2x^2y^2z^2}=1+\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2},$$

$$\therefore\ \frac{(a^3+b^3)^2}{4x^2y^2z^2}-\frac{a^3b^3+c^6}{2x^2y^2z^2}=2.\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+2.\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right),$$

$$=\frac{2b^3}{xyz}-4,\text{ by }(a);$$

$$\therefore\ 4xyz=b^3\pm\sqrt{2c^6-a^6}.$$

Again, from (1), (2), (3), and (4),

$$\left.\begin{aligned}(x^2-y^2)(y^2-z^2)(z^2-x^2)&=a^3b^3,\\(x^2+y^2)(y^2+z^2)(z^2+x^2)&=c^6,\\(x^4+y^4)(y^4+z^4)(z^4+x^4)&=p^{12},\end{aligned}\right\}$$

which are analogous to (1), (2), and (3);

$$\therefore\ 4x^2y^2z^2=c^6\pm\sqrt{2p^{12}-a^6b^6};$$

$$\therefore\ 4c^6\pm4\sqrt{2p^{12}-a^6b^6}=(b^3\pm\sqrt{2c^6-a^6})^2.$$

Ex. 31. Eliminate x from the equations

$$\left.\begin{aligned}32\frac{c}{a}&=\left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\\32\frac{a}{c}&=\left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{By addition,}\quad 32\left(\frac{c}{a}+\frac{a}{c}\right)&=\left(\frac{x}{a}+\frac{a}{x}\right)^5,\\\text{by subtraction,}\quad 32\left(\frac{c}{a}-\frac{a}{c}\right)&=\left(\frac{x}{a}-\frac{a}{x}\right)^5;\end{aligned}\right\}$$

$$\left.\begin{aligned}\therefore\ \frac{x}{a}+\frac{a}{x}&=2\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{1}{5}},\\\frac{x}{a}-\frac{a}{x}&=2\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{1}{5}},\end{aligned}\right\}$$

squaring and subtracting, we have

$$\therefore\ 4=4\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-4\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}};$$

$$\text{or}\ 1=\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}.$$

Ex. 32. Eliminate x, y, z from the equations

$$\left.\begin{aligned}\frac{x}{y}+\frac{y}{z}+\frac{z}{x}&=\alpha,\\\frac{x}{z}+\frac{y}{x}+\frac{z}{y}&=\beta,\\\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)&=\gamma.\end{aligned}\right\}$$

From (1), and (3), we have

$$\left(a-\frac{z}{x}\right)\left(a-\frac{x}{y}\right)\left(a-\frac{y}{z}\right)=\gamma;$$

$$\therefore\ \gamma=a^3-a^2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+a\left(\frac{z}{x}.\frac{x}{y}+\frac{z}{x}.\frac{y}{z}+\frac{x}{y}.\frac{y}{z}\right)-\frac{z}{x}.\frac{x}{y}.\frac{y}{z},$$

$$=a^3-a^2.\alpha+a.\beta-1, \text{ by } (2);$$

$$\therefore\ \alpha\beta=1+\gamma.$$

Ex. 33. A symmetrical form of the condition, that the equations

$$ax+a'=bx+b'=cx+c'$$

may be simultaneous, is

$$a'(b-c)+b'(c-a)+c'(a-b)=0;$$

and a symmetrical form of the value of x is

$$\frac{aa'(b-c)+bb'(c-a)+cc'(a-b)}{(a-b)(b-c)(c-a)}.$$

We have $-x=\dfrac{a'-b'}{a-b}=\dfrac{b'-c'}{b-c}=\dfrac{c'-a'}{c-a}$,

and $\therefore\ =\dfrac{(a'-b')c'+(b'-c')a'+(c'-a')b'}{(a-b)c'+(b-c)a'+(c-a)b'}$; (Art. 195*)

the numerator of this fraction is identically $=0$; therefore the denominator must also $=0$,

$$\text{or } a'(b-c)+b'(c-a)+c'(a-b)=0,$$

the condition sought.

$$\text{Also } -x=\frac{(a'-b')ab+(b'-c')bc+(c'-a')ca}{(a-b)ab+(b-c)bc+(c-a)ca},$$

$$=\frac{aa'(b-c)+bb'(c-a)+cc'(a-b)}{(a-b)ab-c(a^2-b^2)+c^2(a-b)},$$

$$=\frac{aa'(b-c)+bb'(c-a)+cc'(a-b)}{(a-b)\{c-(a+b)c+ab\}};$$

$$\therefore\ x=\frac{aa'(b-c)+bb'(c-a)+cc'(a-b)}{(a-b)(b-c)(c-a)}.$$

Ex. 34. If $\dfrac{a-a'}{a'-a''}=\dfrac{b-b'}{b'-b''}=\dfrac{c-c'}{c'-c''}$; prove that

$$\frac{ab'-a'b}{b'b''-a''b'}=\frac{bc'-b'c}{b'c''-b''c'}=\frac{ca'-c'a}{c'a''-c''a'},$$

and = each of the former; and each of these six fractions

$$=\frac{(a+b+c)-(a'+b'+c')}{(a'+b'+c')-(a''+b''+c'')}.$$

$$\text{Each fraction}=\frac{(a-a')b'-(b-b')a'}{(a'-a'')b'-(b'-b'')a'},$$

$$\text{and}=\frac{(b-b')c'-(c-c')b'}{(b'-b'')c'-(c'-c'')b'},$$

$$\text{and}=\frac{(c-c')a'-(a-a')c'}{(c'-c'')a'-(a'-a'')c'},$$

$$\text{and}=\frac{(a-a')+(b-b')+(c-c')}{(a'-a'')+(b'-b'')+(c'-c'')},$$

$$i.e.=\frac{ab'-a'b}{a'b''-a''b'}=\frac{bc'-b'c}{b'c''-b''c'}=\frac{ca'-c'a}{c'a''-c''a'}$$

$$=\frac{(a+b+c)-(a'+b'+c')}{(a'+b'+c')-(a''+b''+c'')}.$$

Ex. 35. Prove that the six equations

$$a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0,$$
$$b_1(c_2a_3-c_3a_2)+b_2(c_3a_1-c_1a_3)+b_3(c_1a_2-c_2a_1)=0,$$
$$c_1(a_2b_3-a_3b_2)+c_2(a_3b_1-a_1b_3)+c_3(a_1b_2-a_2b_1)=0,$$
$$d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)=0,$$
$$d_1(c_2a_3-c_3a_2)+d_2(c_3a_1-c_1a_3)+d_3(c_1a_2-c_2a_1)=0,$$
$$d_1(a_2b_3-a_3b_2)+d_2(a_3b_1-a_1b_3)+d_3(a_1b_2-a_2b_1)=0,$$

are equivalent to only two independent equations: and find them in the most simple symmetrical shape.

Consider the equations

$$\left.\begin{aligned}a_1x+b_1y+c_1z&=d_1\\a_2x+b_2y+c_2z&=d_2\\a_3x+b_3y+c_3z&=d_3\end{aligned}\right\}\ldots(\text{A}).$$

If y and z be eliminated by cross multiplication, in the result the coefficient of x = left-hand member of 1st of the given equations, and the other term = left-hand member of 4th of the given equations. Whence $x=\frac{0}{0}$. In like manner $y=\frac{0}{0}$, $z=\frac{0}{0}$.

The given equations therefore are the conditions that the equations (A) should be indeterminate; and if we write them in the form

$$\frac{a_1}{d_1}x+\frac{b_1}{d_1}y+\frac{c_1}{d_1}z=1, \ \&c.$$

we shall find by referring to *Comp.* p. 64, Ex. 19, that the conditions are

$$\frac{\frac{a_1}{d_1}-\frac{a_2}{d_2}}{\frac{a_2}{d_2}-\frac{a_3}{d_3}}=\frac{\frac{b_1}{d_1}-\frac{b_2}{d_2}}{\frac{b_2}{d_2}-\frac{b_3}{d_3}}=\frac{\frac{c_1}{d_1}-\frac{c_2}{d_2}}{\frac{c_2}{d_2}-\frac{c_3}{d_3}}.$$

The given equations are therefore equivalent to these two.

Ex. 36. There are p coins $c_1, c_2, c_3 \ldots c_p$. If m_1 of the coins $c_1=n_2$ of c_2; m_2 of $c_2=n_3$ of c_3; m_3 of $c_3=n_4$ of c_4, &c.; m_{p-1} of $c_{p-1}=n_p$ of c_p: how many of $c_p=n_1$ of c_1?

Establish the "Chain Rule."

$$1 \text{ of } c_1=\frac{n_2}{m_1} \text{ of } c_2,$$

$$1 \text{ of } c_2=\frac{n_3}{m_2} \text{ of } c_3,$$

$$1 \text{ of } c_3=\frac{n_4}{m_3} \text{ of } c_4,$$

$$\&\text{c.}=\&\text{c.}$$

$$1 \text{ of } c_{p-1}=\frac{n_p}{m_{p-1}} \text{ of } c_p;$$

$$\therefore 1 \text{ of } c_1=\frac{n_2}{m_1}\cdot\frac{n_3}{m_2}\cdot\frac{n_4}{m_3}\cdots\cdots\frac{n_p}{m_{p-1}} \text{ of } c_p;$$

$$\therefore n_1 \text{ of } c_1=\frac{n_1\cdot n_2\cdot n_3\cdots\cdots n_p}{m_1\cdot m_2\cdot m_3\cdots\cdots m_{p-1}} \text{ of } c_p.$$

The chain rule is; if m_1 things of one kind $=n_2$ of a second, m_2 of these $=n_3$ of a third, &c...., and m_p of the last kind $=n_1$ of the first, then

$$m_1\cdot m_2\cdot m_3\ldots m_p=n_1\cdot n_2\cdot n_3\ldots n_p.$$

This is proved in the above Example.

Ex. 37. There are two amalgams of the same bulk, each composed of mercury and gold, in the ratios of 2 : 9, and 3 : 19, respectively. If they were fused together, what would be the ratio of mercury to gold in the resulting amalgam?

In the second amalgam 3 parts out of 22 are mercury: in the first 2 parts out of 11, *i. e.* 4 out of 22 are mercury. As they are of the same bulk, let each contain 22 units of bulk. Then there are 7 units of mercury altogether, and 44 units in the resulting amalgam; therefore the ratio of mercury to gold is 7 : 37.

Ex. 38. At noon on a certain day a clock and a watch, each of which goes uniformly, are set to true time. It was calculated that the clock would be as much wrong when it should shew any time as the watch would be when it should come to shew the same time; and that it would be midnight by the clock one second before it would be midnight by the watch. Find, in fractions of a second, the daily gaining and losing rates of the clock and the watch.

Let the clock and the watch gain and lose x, y, seconds per day, respectively.

Then 86400 seconds of real time

$$= 86400 + x \text{ seconds of clock time,}$$

$$\text{and } = 86400 - y \text{ seconds of watch time.}$$

Let t be any time indicated by the clock; then the real time is $\frac{86400}{86400+x}.t$; and the clock is wrong by $t - \frac{86400}{86400+x}.t$.

So the watch would be wrong by $\frac{86400}{86400-y}t - t$.

And these, by question, are equal;

$$\therefore \frac{86400}{86400-y} + \frac{86400}{86400+x} = 2.$$

Also because it is midnight by the clock 1 second before it is midnight by the watch,

$$\left(\frac{86400}{86400-y} - \frac{86400}{86400+x}\right) \times 43200 = 1;$$

$$\left.\begin{aligned} \therefore \frac{86400}{86400-y} &= \tfrac{1}{2}\left(2 + \frac{1}{43200}\right) = \frac{86400+1}{86400}, \\ \frac{86400}{86400+x} &= \tfrac{1}{2}\left(2 - \frac{1}{43200}\right) = \frac{86400-1}{86400}; \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore \frac{y}{86400} &= \frac{1}{86400+1}, \\ \frac{x}{86400} &= \frac{1}{86400-1}; \end{aligned}\right\}$$

$$\therefore \text{ the clock gains } \frac{86400}{86400-1} \text{ sec}^s.,$$

$$\text{and the watch loses } \frac{86400}{86400+1} \text{ sec}^s. \text{ per day.}$$

Ex. 39. A shilling's worth of Bavarian kreuzers is more numerous by 6 than a shilling's worth of Austrian kreuzers; and 15 Austrian kreuzers are worth a penny more than 15 Bavarian kreuzers. How many of them respectively make a shilling?

Enunciate the problem indicated by the negative result.

Let x Austrian kreuzers, or $x+6$ Bavarian, $=1$ shilling;

$$\therefore\ 15.\frac{12}{x} = 15.\frac{12}{x+6}+1,$$

$$\frac{1}{x} - \frac{1}{x+6} = \frac{1}{180},$$

$$x^2+6x = 1080;$$

$\therefore\ x = 30$, or -36, the number of Austrian kreuzers in a shilling; and

$x+6 = 36$, or -30, the number of Bavarian kreuzers in a shilling.

To find the problem of which the negative result is the solution, write $-x$ for x: and the original equation becomes

$$15.\frac{12}{x} = 15.\frac{12}{x-6} - 1.$$

Whence we see that the question would be:—"A shilling's worth of Bavarian kreuzers is *less* numerous by 6 than a shilling's worth of Austrian kreuzers; and 15 Austrian kreuzers are worth a penny *less* than 15 Bavarian kreuzers: how many of each make a shilling?" This is the same as the original question in a different shape.

Ex. 40. From a vessel which will contain a gallons, filled with a fluid α_1, a gallon being drawn off, the vessel is filled up with a fluid α_2; a gallon being drawn off from the mixture, the vessel is filled up with a fluid α_3; and so on, until the vessel contains a portion of each of the fluids $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$. How much of each fluid does it contain?

Whenever a gallon is drawn off, the quantity of any fluid in the vessel is $\frac{a-1}{a}$, or $1-\frac{1}{a}$, of what it was before;

therefore, after the first operation, the vessel contains

$$\left(1-\frac{1}{a}\right)a \text{ gallons of } a_1, \text{ and 1 gallon of } a_2;$$

after the second, it contains

$$\left(1-\frac{1}{a}\right)\left(1-\frac{1}{a}\right)a \text{ of } a_1, \quad 1-\frac{1}{a} \text{ of } a_2, \text{ and 1 of } a_3;$$

after the third, it contains

$$\left(1-\frac{1}{a}\right)^3 a \text{ of } a_1, \left(1-\frac{1}{a}\right)^2 \text{ of } a_2, \ 1-\frac{1}{a} \text{ of } a_3, \text{ and 1 of } a_4; \text{ and so on.}$$

$\therefore$ after the last, it contains

$$\left(1-\frac{1}{a}\right)^{n-1} a \text{ of } a_1, \left(1-\frac{1}{a}\right)^{n-2} \text{ of } a_2, \left(1-\frac{1}{a}\right)^{n-3} \text{ of } a_3,$$

$$\ldots\left(1-\frac{1}{a}\right) \text{ of } a_{n-1}, \text{ and 1 of } a_n.$$

Ex. 41. Prove that

$$(a+b)(b+c)(c+a) > 8abc, \text{ and } < \frac{8}{3}(a^3+b^3+c^3),$$

unless $a=b=c$.

(1) $(a+b)(b+c)(c+a) \gtreqless 8abc,$

according as $a^2b+ab^2+a^2c+ac^2+b^2c+bc^2 \gtreqless 6abc$;

i.e. as $a(b^2-2bc+c^2)+b(c^2-2ca+a^2)+c(a^2-2ab+b^2) \gtreqless 0$;

...... $a(b-c)^2+b(c-a)^2+c(a-b)^2 \gtreqless 0.$

But $a(b-c)^2+b(c-a)^2+c(a-b)^2>0$, unless $a=b=c$;

$\therefore$ $(a+b)(b+c)(c+a)>8abc$, unless $a=b=c$.

(2) $(a+b)(b+c)(c+a) \gtreqless \frac{8}{3}(a^3+b^3+c^3),$

according as

$$3(a+b)(b+c)(c+a) \gtreqless 8\{(a+b+c)^3-3(a+b)(b+c)(c+a)\};$$

i.e. as $27(a+b)(b+c)(c+a) \gtreqless 8(a+b+c)^3$,

$$\text{or } \{2(a+b+c)\}^3,$$

$$\text{or } (a+b+b+c+c+a)^3;$$

i.e. as $\sqrt{(a+b)(b+c)(c+a)} \gtreqless \dfrac{(a+b)+(b+c)+(c+a)}{3}$.

But, by Art. 221, Ex. 6,

$$\frac{(a+b)+(b+c)+(c+a)}{3} > \sqrt[3]{(a+b)(b+c)(c+a)},$$

except when $a+b$, $b+c$, $c+a$ are equal;

$$\therefore\ (a+b)(b+c)(c+a) < \tfrac{8}{3}(a^3+b^3+c^3), \text{ unless } a=b=c.$$

Ex. 42. Which is greater,

$$mx+\frac{n}{x}, \text{ or } m+n? \quad mx^n+\frac{n}{x^m}, \text{ or } m+n?$$

$$(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d), \text{ or } 16abcd?$$

(1) $\left(mx+\dfrac{n}{x}\right)-(m+n)=m(x-1)+\dfrac{n}{x}(1-x)$,

$$=\left(m-\frac{n}{x}\right)(x-1);$$

$\therefore$ when $x>1$, the former, or the latter, of the given expressions is the greater, according as $m-\dfrac{n}{x}$ is positive or negative;

$$i.e. \text{ according as } x \gtrless \frac{n}{m}.$$

So when $x<1$, the former, or the latter, is the greater, according as $x \lessgtr \dfrac{n}{m}$.

(2) $a_1+a_2+a_3+\ldots+a_{m+n}>(m+n)(a_1.a_2.a_3\ldots a_{m+n})^{\frac{1}{m+n}}$. (Art. 221).

Let $a_1=a_2=a_3=\ldots=a_m=x^n$.

And $a_{m+1}=a_{m+2}=a_{m+3}=\ldots=a_{m+n}=x^{-m}$;

$$\therefore\ mx^n+\frac{n}{x^m}>(m+n)(x^{mn}.x^{-mn})^{\frac{1}{m+n}},$$

$$>m+n,$$

where m and n are positive integers.

If m and n be fractional, let them $=\frac{p}{q}$, $\frac{r}{s}$; where p, q, r, s, are positive integers.

$$\text{Then } \frac{p}{q}x^{\frac{r}{s}}+\frac{r}{s}\cdot\frac{1}{x^{\frac{p}{q}}} \gtrless \frac{p}{q}+\frac{r}{s},$$

$$\text{according as } ps.x^{\frac{r}{s}}+qr\frac{1}{x^{\frac{p}{q}}} \gtrless ps+qr;$$

$$i.e. \text{ as } ps.y^{qr}+qr.\frac{1}{y^{ps}} \gtrless ps+qr, \text{ putting } y \text{ for } x^{\frac{1}{qs}}.$$

But, by the former case,

$$ps.y^{qr}+qr.\frac{1}{y^{ps}} > ps+qr;$$

$$\therefore\ mx^n+\frac{n}{x^m} > m+n, \text{ if } m \text{ and } n \text{ be positive.}$$

(3) The expressions here being symmetrical, we can without losing any quantity suppose the letters arranged in descending order of magnitude.

From Ex. 24, p. 223, we have

$$(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$$
$$+(a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)=16abcd;$$

therefore the former, or the latter, of the given expressions is the greater, according as the second of the above continued products is negative or positive.

Now $a>b$, $b>c$, $c>d$;

$\therefore\ a+b-c-d$, and $a-b+c-d$, are both positive;

therefore the former, or the latter, of the given expressions is the greater, according as $a-b-c+d$ is negative or positive;

$$i.e. \text{ as } a+d \text{ is } \lessgtr b+c,$$

or as the greatest and least of a, b, c, d together is $\lessgtr$ the other two together.

Ex. 43. If a, b, c, be in Harmonical Progression, prove that

$$\frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b}$$

are also in Harmonical Progression.

We have $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ in A.P.

$$\therefore \frac{a+b+c}{a},\ \frac{a+b+c}{b},\ \frac{a+b+c}{c} \text{ are in A.P.;}$$

$$\therefore \frac{a+b+c}{a}-1,\ \frac{a+b+c}{b}-1,\ \frac{a+b+c}{c}-1 \text{ are in A.P.,}$$

$$\text{or } \frac{b+c}{a},\ \frac{c+a}{b},\ \frac{a+b}{c} \text{ are in A.P.;}$$

$$\therefore \frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b} \text{ are in H.P.}$$

Ex. 44. The relative powers of the instruments $a_1, a_2, a_3, \ldots a_n$ are expressed by saying that generally $a_1, a_2, a_3, \ldots a_{r-1}$ working together can do as much work in one day as a_r can do in p days. If a_n can do a piece of work in one hour, how long will it take a_1 and a_r, where $r>1$, to do it?

Let a_1 do the work in x hours;

$$\therefore \text{ in 1 hour } a_1 \text{ does } \frac{1}{x}\text{th of it;}$$

$$\ldots\ldots\ldots\ldots\ldots\ a_2 \ldots\ldots \frac{1}{px} \ldots\ldots\ldots;$$

$$\ldots\ldots\ldots\ldots a_1 \text{ and } a_2 \text{ together do } \left(1+\frac{1}{p}\right).\frac{1}{x} \text{ of it;}$$

$$\ldots\ldots\ldots\ldots\ldots\ a_3 \text{ does } \frac{1}{p}\left(1+\frac{1}{p}\right).\frac{1}{x} \text{ of it;}$$

$$\ldots\ldots\ldots\ldots\ldots a_1, a_2, a_3 \text{ together do } \left(1+\frac{1}{p}\right)^2.\frac{1}{x} \text{ of it;}$$

$$\ldots\ldots\ldots\ldots\ldots\ a_4 \text{ does } \frac{1}{p}\left(1+\frac{1}{p}\right)^2.\frac{1}{x} \text{ of it;}$$

&c.

$$\ldots\ldots\ldots\ldots\ldots\ a_n \ldots\ldots \frac{1}{p}\left(1+\frac{1}{p}\right)^{n-2}.\frac{1}{x} \ldots\ldots$$

But a_n does the work in 1 hour;

$$\therefore \frac{1}{p}\left(1+\frac{1}{p}\right)^{n-2}.\frac{1}{x}=1, \text{ or } x=\frac{1}{p}\left(1+\frac{1}{p}\right)^{n-2},$$

the number of hours in which a_1 does it.

In 1 hour a_r does $\frac{1}{p}\left(1+\frac{1}{p}\right)^{r-s}\cdot\frac{1}{x}$ of the work,

$$i.e.\ \frac{\frac{1}{p}\left(1+\frac{1}{p}\right)^{r-s}}{\frac{1}{p}\left(1+\frac{1}{p}\right)^{n-s}} \text{ of it.}$$

$$\therefore\ a_r \text{ does the whole in } \left(1+\frac{1}{p}\right)^{n-r} \text{ hours.}$$

Ex. 45. If m and n be any two positive integers of which m is the greater, and $p=2^m$, $q=2^n$; prove that $x^{2p}+a^px^p+a^{2p}$ is exactly divisible by both $x^{2q}+a^qx^q+a^{2q}$ and $x^{2q}-a^qx^q+a^{2q}$.

We have
$$(x^2+ax+a^2)(x^2-ax+a^2)=x^4+a^2x^2+a^4,$$
$$(x^4+a^2x^2+a^4)(x^4-a^2x^2+a^4)=x^8+a^4x^4+a^8,$$
&c.

$\therefore$ both x^2+ax+a^2, and x^2-ax+a^2, divide exactly $x^{2r}+a^rx^r+a^{2r}$, where the index r is obtained by successive duplication of the preceding indices.

If then we write x^q, a^q, for x and a in the above (where $q=2^n$), p, or 2^m, is obtained from q, or 2^n, by successive duplication;

$\therefore$ $x^{2q}+a^qx^q+a^{2q}$, and $x^{2q}-a^qx^q+a^{2q}$, divide $x^{2p}+a^px^p+a^{2p}$ without remainder.

Ex. 46. If $a_r=\dfrac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}$, prove that

$$a_1-\frac{a_2}{2}+\frac{a_3}{3}-\ldots+\frac{(-1)^{n-1}}{n}=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}.$$

Let
$$f(n)=a_1-\frac{a_2}{2}+\frac{a_3}{3}-\ldots+\frac{(-1)^{n-1}}{n},$$
$$=n-\frac{n(n-1)}{1\cdot 2}\cdot\tfrac{1}{2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\tfrac{1}{3}-\&\text{c.}\ldots$$
$$+n\frac{(-1)^{n-2}}{n-1}+\frac{(-1)^{n-1}}{n};$$

$$\therefore f(n-1)=(n-1)-\frac{(n-1)(n-2)}{1\cdot 2}\cdot\tfrac{1}{2}+\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}\cdot\tfrac{1}{3}-\&\text{c.}\ldots$$
$$+\frac{(-1)^{n-2}}{n-1};$$

$$\therefore f(n)-f(n-1)=1-\frac{n-1}{1.2}+\frac{(n-1)(n-2)}{1.2.3}-\&c.\ldots+(-1)^{n-2}+\frac{(-1)^{n-1}}{n},$$

$$=\frac{1}{n}\left\{\begin{matrix} n-\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}-\&c.\ldots \\ +n(-1)^{n-2}+(-1)^{n-1}\end{matrix}\right\},$$

$$=\frac{1}{n}\left[1-\left\{\begin{matrix}1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\&c.\ldots \\ +n(-1)^{n-1}+(-1)^{n}\end{matrix}\right\}\right],$$

$$=\frac{1}{n}\{1-(1-1)^n\}=\frac{1}{n}.$$

$$\text{So } f(n-1)-f(n-2)=\frac{1}{n-1};$$

$$f(n-2)-f(n-3)=\frac{1}{n-2},$$

$$\&c.=\&c.$$

$$f(2)-f(1)=\tfrac{1}{2},$$

$$\text{also } f(1)=1;$$

$\therefore$ by addition, $f(n)=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

Ex. 47. If $p_r=\dfrac{1.3.5\ldots(2r-1)}{2.4.6\ldots\ 2r}$, prove that

$$p_{2n+1}+p_1.p_{2n}+p_2.p_{2n-1}+\ldots+p_{n+1}.p_{n-2}+p_n.p_{n+1}=\tfrac{1}{2}.$$

We have $p_r=\dfrac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots(r-\frac{1}{2})}{1.2.3\ldots\ r}=$ coefficient of x^r in the expansion of $(1-x)^{-\frac{1}{2}}$.

$$\therefore (1-x)^{-\frac{1}{2}}=1+p_1x+p_2x^2\ldots+p_nx^n+p_{n+1}x^{n+1}+\ldots$$
$$+p_{2n}x^{2n}+p_{2n+1}x^{2n+1}+\&c.;$$

$\therefore$ multiplying this by itself, we have

$$(1-x)^{-1}=\{1+p_1x+p_2x^2+\ldots+p_nx^n+p_{n+1}x^{n+1}+\ldots$$
$$+p_{2n}x^{2n}+p_{2n+1}x^{2n+1}+\ldots\}$$
$$\times\{1+p_1x+p_2x^2+\ldots+p_nx^n+p_{n+1}x^{n+1}+\ldots+p_{2n}x^{2n}+p_{2n+1}x^{2n+1}+\ldots\}$$

Equating the coefficients of x^{2n+1}, we have

$$1=p_{2n+1}+p_1.p_{2n}+p_2.p_{2n-1}+\ldots+p_n.p_{n+1}+p_{n+1}.p_n+\ldots+p_1.p_{2n}+p_{2n+1};$$

$$\therefore p_{2n+1}+p_1.p_{2n}+p_2.p_{2n-1}+\ldots+p_n.p_{n+1}=\tfrac{1}{2}.$$

Ex. 48. If t_0, t_1, t_2, t_3, &c., represent the terms of the binomial expansion of $(a+x)^n$, prove that

$$(t_0-t_2+t_4-\&c.)^2+(t_1-t_3+t_5-\&c.)^2=(a^2+x^2)^n.$$

$(t_0-t_2+t_4-\&c.)^2+(t_1-t_3+t_5-\&c.)^2$

$$=\{(t_0-t_2+t_4-\ldots)+\sqrt{-1}(t_1-t_3+t_5-\ldots)\}$$
$$\times\{(t_0-t_2+t_4-\ldots)-\sqrt{-1}\,(t\ -t_3+t_5-\ldots)\},$$
$$=(t_0+t_1\sqrt{-1}-t_2-t_3\sqrt{-1}+t_4+\ldots)(t_0-t\ \sqrt{-1}-t_2+t_3\sqrt{-1}+t_4-\ldots)$$
$$=(a+x\sqrt{-1})^n(a-x\sqrt{-1})^n=(a^2+x^2)^n.$$

Ex. 49. If a_0, a_1, a_2, a_3, &c. be the coefficients in order of the expansion of $(1+x+x^2+\ldots+x^p)^n$, prove that

(1) their sum $=(p+1)^n$;

(2) $a_1+2a_2+3a_3+\ldots+np\,.\,a_{np}=\frac{1}{2}np(p+1)^n$.

We shall have $a_0+a_1x+a_2x^2+a_3x\ +\ldots.+a_{np}x^{np}$

$$=(1+x+x^2+\ldots+x^p)^n,\ \text{identically}.$$

For x write $1+y$; then

$(a_0+a_1+a_2+\ldots+a_{np})+(a_1+2a_2+3a_3+\ldots+np\,.\,a_{np})y+$ terms involving higher powers of y,

$$=\Big\{(p+1)+(1+2+3+\ldots+p)y+\text{terms involving higher powers of } y\Big\}^n$$

$$=\{(p+1)+\frac{(p+1)p}{2}\,.\,y+\ldots\}^n,$$

$$=(p+1)^n+n\,(p+1)^{n-1}\,.\left\{\frac{(p+1)p}{2}\,.\,y+\ldots\right\}$$
$$+\frac{n\,(n-1)}{2}\,(p+1)^{n-2}\left\{\frac{(p+1)p}{2}\,.\,y+\ldots\right\}^2+\ \&c.$$

$$=(p+1)^n+\frac{np\,(p+1)^n}{2}\,y+\text{terms involving higher powers of } y.$$

This is satisfied by all values of y; therefore equating the first two terms on each side,

$$a_0+a_1+a_2+\ldots+a_{np}=(p+1)^n,$$
$$\text{and}\ \ a_1+2a_2+3a_3+\ldots+np\,.\,a_{np}=\tfrac{1}{2}\,np\,(p+1)^n.^*$$

* The first case might more easily be proved thus:—All the terms are positive, and $\therefore$ the sum of the coefficients will be found by putting $x=1$;

$$\therefore \text{the sum}=(1+1+1^2+\ldots+1^p)^n=(p+1)^n.$$

Ex. 50. Apply the general term in the Multinomial Theorem to prove that the coefficient of x^{2p+1} in the expansion of

$$(a_0+a_1x+a_2x^2+\dots)^2$$

$$\text{is } 2.(a_0.a_{2p+1}+a_1.a_{2p}+a_2.a_{2p-1}+\dots+a_p.a_{p+1}).$$

In the expansion of $(a_0+a_1x+a_2x^2+\dots)^2$ the term involving x^{2p+1} is

$$\lfloor\underline{2}\,.\,\Sigma\frac{a_0^{\alpha_0}}{\lfloor\underline{\alpha_0}}\cdot\frac{a_1^{\alpha_1}}{\lfloor\underline{\alpha_1}}\cdot\frac{a_2^{\alpha_2}}{\lfloor\underline{\alpha_2}}\dots x^{\alpha_1+2\alpha_2+3\alpha_3+\dots}$$

$$\left.\begin{aligned}\text{where } \alpha_0+\alpha_1+\alpha_2+\alpha_3+\dots&=2,\\ \alpha_1+2\alpha_2+3\alpha_3+\dots&=2p+1.\end{aligned}\right\}$$

The solutions of these will be, (1) when one of the α's $=2$, and all the others $=0$; (2) when two of them $=1$, and the others $=0$.

And in case (1) the second equation is not satisfied, as the right-hand side is odd.

The solutions then are

$$\begin{aligned}\alpha_0&=1, & \alpha_{2p+1}&=1,\\ \alpha_1&=1, & \alpha_{2p}&=1,\\ \alpha_2&=1, & \alpha_{2p-1}&=1,\\ \alpha_3&=1, & \alpha_{2p-2}&=1,\\ &\&c.\\ \alpha_p&=1, & \alpha_{p+1}&=1;\end{aligned}$$

$\therefore$ the coefficient of x^{2p+1} will be

$$2.\{a_0.a_{2p+1}+a_1a_{2p}+a_2.a_{2p-1}+\dots+a_p.a_{p+1}\}.$$

Ex. 51. In the expansion of $(1+x+x^2+\dots+x^r)^{2n}$, where n is a positive integer, prove that, (1) the coefficients of terms equidistant from the beginning and the end are equal; (2) the coefficient of the middle term is the greatest; (3) the coefficients continually increase from the first up to the greatest.

(1) Let

$$(1+x+x^2+\dots+x^r)^{2n}=a_0+a_1x+a_2x^2+\dots+a_{2nr-1}x^{2nr-1}+a_{2nr}x^{2nr}.$$

Then

$$\left(1+\frac{1}{x}+\frac{1}{x^2}+\dots+\frac{1}{x^r}\right)^{2n}=a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots+\frac{a_{2nr-1}}{x^{2nr-1}}+\frac{a_{2nr}}{x^{2nr}};$$

$$\therefore\ (x^r+x^{r-1}+\dots+x+1)^{2n}=a_0x^{2nr}+a_1x^{2nr-1}+a_2x^{2nr-2}+\dots+a_{2nr-1}x+a_{2nr}.$$

Comparing this with the original expression, we have

$$a_0 = a_{2nr},\ a_1 = a_{2nr-1},\ a_2 = a_{2nr-2},\ \text{\&c.};\ \text{Q.E.D.}$$

(2) and (3) By actual multiplication,

$$(1+x+x^2+\ldots+x^r)^2 = 1+2x+3x^2+\ldots+rx^{r-1}+(r+1)x^r+rx^{r+1}+\ldots$$
$$+2x^{2r-1}+x^{2r};$$

therefore in this case the law is true.

Suppose the law to hold when the index is $2n$, and let $\rho = nr$, and consequently the coefficient of x^ρ the greatest: and let

$$(1+x+x^2+\ldots+x^r)^{2n} = 1+a_1x+a_2x^2+\ldots+a_{\rho-1}x^{\rho-1}+a_\rho x^\rho+a_{\rho-1}x^{\rho+1}+\ldots$$

Now

$$(1+x+x^2+\ldots+x^r)^2 = 1+2x+3x^2+\ldots+rx^{r-1}+(r+1)x^r+rx^{r+1}+\ldots$$
$$+2x^{2r-1}+x^{2r}.$$

Multiplying, and observing, that the term involving $x^{\rho+r}$ is the middle term of the expansion of

$$(1+x+x^2+\ldots+x^r)^{2n+2},\ \text{we have}$$

coefficient of $x^{\rho+r} = a_{\rho+r}+2a_{\rho+r-1}+3a_{\rho+r-2}+\ldots+ra_{\rho+1}+(r+1)a_\rho+ra_{\rho-1}+\ldots$
$$+2a_{\rho-r+1}+a_{\rho-r},$$

coefficient of $x^{\rho+r-1} = a_{\rho+r-1}+2a_{\rho+r-2}+3a_{\rho+r-3}+\ldots+(r-1)a_{\rho+1}+ra_\rho$
$$+(r+1)a_{\rho-1}+\ldots+2a_{\rho-r}+a_{\rho-r-1},$$

$\therefore$ coefficient of $x^{\rho+r}$ − coefficient of $x^{\rho+r-1}$

$$= a_{\rho+r}+a_{\rho+r-1}+a_{\rho+r-2}+\ldots+a_{\rho+1}+a_\rho$$
$$-a_{\rho-r}-a_{\rho-r+1}-a_{\rho-r+2}-\ldots-a_{\rho-1}-a_{\rho-r-1},$$
$$= a_\rho - a_{\rho-r-1},$$
$$> 0.$$

In like manner,

coefficient of $x^{\rho+r-1}$ − coefficient of $x^{\rho+r-s-1}$

$$= a_{\rho+r-s}+a_{\rho+r-s-1}+a_{\rho+r-s-2}+\ldots+a_{\rho-s+1}+a_{\rho-s}$$
$$-a_{\rho-r-s-1}-a_{\rho-r-s}-a_{\rho-r-s-1}-\ldots-a_{\rho-s-2}-a_{\rho-s-1}.$$

Now every term of the upper line is greater than the corresponding term of the lower line: for

general term of upper line $= a_{\rho+r-s-r'}$ where r' is not $> r$,
and corresponding .... lower ..... $= a_{\rho-r-s+r'-1}$.

i. Let $r-r'>s$, then $a_{\rho+r-s-r'}-a_{\rho-r-s+r'-1}$

$$=a_{\rho+(r-s-r')}-a_{\rho+(r+s-r'+1)}>0;$$

$$\because\ r-s-r'<r+s-r'+1.$$

ii. Let $r-r'<s$, then $a_{\rho+r-s-r'}-a_{\rho-r-s+r'-1}$

$$=a_{\rho-\{s-(r-r')\}}-a_{\rho-(s+r-r'+1)}>0;$$

$$\because\ s-(r-r')<s+r-r'+1;$$

therefore coefficient of $x^{\rho+r-s}>$ coefficient of $x^{\rho+r-s-1}$ for all values of s.

Hence the coefficient of $x^{\rho+r}$, *i.e.* the coefficient of the middle term is greater than the coefficient of any preceding term, and therefore greater than the coefficient of any succeeding term: also the coefficients continually increase from the first up to the greatest.

That is, if the assumed law hold when the index is $2n$, it holds when the index is $2n+2$. But it holds when the index is 2, therefore when the index is 4, and therefore generally.

Ex. 52. If a_0, a_1, a_2, a_3, &c. be the coefficients in order of the expansion of $(1+x+x^2)^n$, prove that

$$a_0^2-a_1^2+a_2^2-a_3^2+\ldots+(-1)^{n-1}a_{n-1}^2=\tfrac{1}{2}a_n\{1-(-1)^n a_n\}.$$

$$(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n+a_{n-1}x^{n+1}+\ldots$$
$$+a_1x^{2n-1}+a_0x^{2n};$$

for the coefficients of terms equidistant from the beginning and the end are equal.

So, writing $-x$ for x,

$$(1-x+x^2)^n=a_0-a_1x+a_2x^2-\ldots+a_{n-1}(-x)^{n-1}+a_n(-x)^n+a_{n-1}(-x)^{n+1}+\ldots$$
$$-a_1x^{2n-1}+a_0x^{2n};$$

$$\therefore\ (1+x^2+x^4)^n=\text{the product of the above two series.}$$

Also $(1+x^2+x^4)^n=a_0+a_1x^2+a_2x^4+\ldots+a_{n-1}x^{2n-2}+a_nx^{2n}+\ldots+a_0x^{4n}$;

therefore performing the multiplication, and equating coefficients of x^{2n},

$$a_n=a_0^2-a_1^2+a_2^2-\ldots+(-1)^{n-1}a_{n-1}^2+(-1)^na_n^2+(-1)^{n-1}a_{n-1}^2+\ldots$$
$$+a_2^2-a_1^2+a_0^2;$$

$$\therefore\ a_n-(-1)^{n-1}a_n^2=a_0^2-a_1^2+a_2^2-\ldots+(-1)^{n-1}a_{n-1}^2+(-1)^{n-1}a_{n-1}^2+\ldots$$
$$-a_1^2+a_0^2;$$

$$\therefore\ a_0^2-a_1^2+a_2^2-\ldots+(-1)^{n-1}a_{n-1}^2=\tfrac{1}{2}\{a_n-(-1)^{n-1}a_n^2\},$$
$$=\tfrac{1}{2}a_n\{1-(-1)^{n-1}a_n\}.$$

Ex. 53. Shew that, if m be not less than n, the greatest coefficient in the expansion of $(a_1+a_2+a_3+\ldots+a_m)^n$ is $\underline{|n}$; and that the number of terms which have this coefficient is $\dfrac{\underline{|m}}{\underline{|n}\,\underline{|m-n}}$.

It is evident, that no two terms in the expansion can combine into one by involving the same combination of powers of a_1, a_2, &c.;

therefore any one of the coefficients can be represented by

$$\frac{\underline{|n}}{\underline{|\alpha_1}\cdot\underline{|\alpha_2}\cdot\underline{|\alpha_3}\ldots\underline{|\alpha_m}},$$

where $\alpha_1, \alpha_2, \ldots \alpha_m$ are all positive integers, and their sum $= n$.

The greatest coefficient therefore will be that in which the denominator of the above fraction $=1$ (provided that it be possible for that denominator to take the value 1), and then it will be $\underline{|n}$.

The denominator can be made equal to 1 by putting α_1, α_2, &c. $=1$, or 0, consistently with the equation

$$\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_m=n\ldots\ldots\ldots\ldots(\text{A}).$$

And when $m \overset{=}{>} n$, these values are admissible;

$\therefore$ when $m \overset{=}{>} n$, the greatest coefficient in the given expansion $=\underline{|n}$.

From (A) we see that n of the α's $=1$, and the rest $=0$.

And the number of terms that we can obtain with the coefficient in question will be the number of different ways in which we can give the value 1 to n of the m α's in equation (A), *i. e.* the number of combinations we can form out of them n together, and therefore is $\dfrac{\underline{|m}}{\underline{|n}\cdot\underline{|m-n}}$.

Ex. 54. Prove by the method of Demonstrative Induction that the number of different throws which can be made with n dice, on the supposition that the sum of the numbers turned up is r, is equal to the coefficient of x^r in the expansion of

$$(x+x^2+x^3+x^4+x^5+x^6)^n.$$

Enunciate the corresponding proposition when each die has p faces, and the numbers marked on the p faces of each die are $a_1, a_2, a_3 \dots a_p$.

Let the throws 1, 2, 3, 4, 5, 6 be respectively represented in each die by x, x^2, x^3, x^4, x^5, x^6.

First let there be two dice: then in the multiplication

$$(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6),$$

the separate terms representing the possible throws of the two dice respectively, we see that any sum, *e. g.* 8, can be thrown in as many ways as x^8 will appear in the above product; *i. e.* the number of ways in which two dice can turn up a sum r = coefficient of x^r in the expansion of

$$(x+x^2+x^3+x^4+x^5+x^6)^2.$$

Suppose this law to hold for n dice, so that number of throws which give the sum of the numbers thrown r = coefficient of x^r in

$$(x+x^2+x^3+x^4+x^5+x^6)^n, \ = a_r, \text{ suppose.}$$

Then from

$$(a_1x+a_2x^2+a_3x^3 \dots +a_rx^r+\dots\dots)(x+x^2+x^3+x^4+x^5+x^6)$$

we see, that coefficient of x^r in

$$(x+x^2+x^3+x^4+x^5+x^6)^{n+1}$$
$$= a_{r-1}+a_{r-2}+a_{r-3}+a_{r-4}+a_{r-5}+a_{r-6}.$$

And number of ways in which the sum turned up with $n+1$ dice is r.

= throw 1 of the last die combined with all the throws of n dice which turn up $r-1$, in No. a_{r-1},

\+ ...... 2 .... $r-2$, ... a_{r-2},

\+ ...... 3 .... $r-3$, ... a_{r-3},

\+ ...... 4 .... $r-4$, ... a_{r-4},

\+ ...... 5 .... $r-5$, ... a_{r-5},

\+ ...... 6 .... $r-6$, ... a_{r-6},

$$= a_{r-1}+a_{r-2}+a_{r-3}+a_{r-4}+a_{r-5}+a_{r-6},$$
$$= \text{coefficient of } x^r \text{ in } (x+x^2+x^3+x^4+x^5+x^6)^{n+1};$$

therefore if the law hold for n dice, it holds for $n+1$. But if holds for 2, and therefore it holds generally.

Cor. If each die had p faces marked with the numbers $a_1, a_2, a_3 \dots a_p$, the number of throws would be the coefficient of x^r in $(x^{a_1}+x^{a_2}+x^{a_3}+\dots+x^{a_p})^n$.

Ex. 55. Prove that the numerators of any two consecutive convergents to a continued fraction are prime to each other, as also their denominators.

This is true for the first 2 or 3 convergents: let it hold good for 2 successive convergents $\frac{N_1}{D_1}$, $\frac{N_2}{D_2}$.

The next one is $\frac{N_3}{D_3}$ where $N_3 = q_3 N_2 + N_1$, $D_3 = q_3 D_2 + D_1$,

q_3 being the corresponding quotient.

If then N_3, N_2 have a common measure, it will also measure $N_3 - q_3 N_2$, or N_1; *i. e.* it is a common measure of N_1 and N_2, which is impossible; therefore N_2, N_3 are prime to each other. The same reasoning will apply to the denominators; therefore if the proposition is true as far as any given convergent, it will be also true as far as the next. But it is true for the first 2 or 3; therefore it is generally true.

Ex. 56. The Cambridge Lent term always ends on a Friday, suppose at midnight. Prove that if it commence on the $2p^{\text{th}}$ day of the week, it will divide either on the $(p-1)^{\text{th}}$ day of the week at midnight, or on the $(3+p)^{\text{th}}$ day of the week at noon: but if it commence on the $(2p+1)^{\text{th}}$ day of the week, it will divide either on the p^{th} day of the week at noon, or on the $(3+p)^{\text{th}}$ day of the week at midnight.

Let the term commence on the r^{th} day of the week. If it had commenced on the Saturday before it did, there would have been an exact number of weeks in it, i. e. r days are wanting to make up an exact number of weeks in the term; therefore the number of days in the term will be of the form $7n-r$, and therefore number of days in the half term $= \frac{1}{2}(7n-r)$; therefore number of days between the beginning of the week in which the term commenced and the division

$$= \tfrac{1}{2}(7n-r) + (r-1) = \tfrac{1}{2}(7n+r) - 1,$$

$$= 7m + \tfrac{1}{2}(r-2), \text{ or } 7m + \tfrac{1}{2}(r+5),$$

according as $n = 2m$ or $2m+1$;

i.e. the term divides on the $\frac{1}{2}(r-2)^{\text{th}}$, or $\frac{1}{2}(r+5)^{\text{th}}$, day of the week.

Let $r = 2p$; then

$$\tfrac{1}{2}(r-2) = p-1, \quad \tfrac{1}{2}(r+5) = 3+p-\tfrac{1}{2};$$

therefore term divides on the $(p-1)^{\text{th}}$ day of the week at midnight, or on the $(3+p)^{\text{th}}$ day of the week at noon.

Let $r=2p+1$; then

$$\tfrac{1}{2}(r-2)=p-\tfrac{1}{2},\quad \tfrac{1}{2}(r+5)=3+p;$$

therefore term divides on the p^{th} day of the week at noon, or on the $(3+p)^{\text{th}}$ day of the week at midnight.

Ex. 57. Prove that

(1) $1+3+5+\ldots\ldots+(2n-1)=n^2$;

(2) $1^3+2^3+3^3+\ldots\ldots+n^3=(1+2+3+\ldots\ldots+n)^2$.

Apply (1) to find solutions in positive integers of $x^n=y^2-z^2$, where n is a positive integer; and apply (2) to find solutions in positive integers of $x^3=y^2-z^2$.

(1) $1+3+5+\ldots+(2n-1)=\{1+(2n-1)\}\dfrac{n}{2}=n^2$.

(2) is proved in Ex. 7, of Art. 342, or may be proved by demonstrative induction, thus:—

$$1^3+2^3=9=(1+2)^2.$$

Let $$1^3+2^3+\ldots+n^3=(1+2+\ldots+n)^2.$$

Then $$1^3+2^3+\ldots+n^3+(n+1)^3=\left\{\frac{n(n+1)}{2}\right\}^2+(n+1)^3,$$

$$=(n+1)^2.\left(\frac{n^2}{4}+n+1\right),$$

$$=(n+1)^2.\left(\frac{n+2}{2}\right)^2,$$

$$=\{1+2+\ldots+(n+1)\}^2;$$

$\therefore$ if the law holds for n, it holds for $n+1$; but it does for 2, and therefore it holds generally.

To obtain a solution of $x^n=y^2-z^2$. Let x be any odd number; then x^n is odd, $=2m+1$.

also $2m+1=\{1+3+5+\ldots+(2m+1)\}-\{1+3+5+\ldots+(2m-1)\}$,

$$=(m+1)^2-m^2;$$

$$\therefore\ y=m+1,\qquad z=m.$$

To obtain a solution of $x^3=y^2-z^2$. Let x be any integer; then

$$y^2-z^2=x^3=(1^3+2^3+3^3+\ldots+x^3)-\{1^3+2^3+\ldots+(x-1)^3\},$$

$$=(1+2+3+\ldots+x)^2-\{1+2+\ldots+(x-1)\}^2;$$

$$\therefore\ y=\frac{x(x+1)}{2},\qquad z=\frac{x(x-1)}{2}.$$

Ex. 58. Prove that the product of any r consecutive integers is divisible by

$$\underline{|a_1}.\underline{|a_2}.\underline{|a_3}\ldots\underline{|a_p}, \text{ if } a_1+a_2+a_3+\ldots\ldots a_p=r.$$

The coefficient of the general term of the expansion of

$$(a_1+a_2+\ldots+a_p)^r$$

is $\dfrac{\underline{|r}}{\underline{|a_1}.\underline{|a_2}.\underline{|a_3}\ldots\underline{|a_p}}$, where $a_1+a_2+a_3+\ldots+a_p=r$,

and must evidently be an integer,

i. e. $\underline{|a_1}.\underline{|a_2}.\underline{|a_3}\ldots\underline{|a_p}=\dfrac{\underline{|r}}{P}$, where P is an integer;

$\therefore$ product of any r consecutive integers $\div(\underline{|a_1}.\underline{|a_2}\ldots\underline{|a_p})$

$=$ (product of the same integers $\div\underline{|r})\times P$

$=$ an integer $\times P$;

$\therefore$ product of any r consecutive integers is divisible by

$$\underline{|a_1}.\underline{|a_2}.\underline{|a_3}\ldots\underline{|a_p}, \text{ if } a_1+a_2+\ldots+a_p=r.$$

Ex. 59. In how many ways can a line 100800 inches long be divided into equal parts, each some multiple of an inch?

$$100800=2^6.3^2.5^2.7;$$

$\therefore$ the whole number of divisors of it is $(6+1)(2+1)(2+1)(1+1)$, including 1 and the given number;

$\therefore$ the number of ways required will $=$ the number of divisors of 100800, not including itself and 1,

and $\therefore\ =7.3.3.2-2=124$.

Ex. 60. How many different rectangular parallelopipeds are there satisfying the condition that each edge of each parallelopiped shall be equal to some one of n given lines all of different lengths?

As the edges may be equal or not, the number required will

$=$ the number of homogeneous products of n things of 3 dimensions,

i. e. $\dfrac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$.

Ex. 61. Prove that if, in any scale of notation, the sum of two numbers is a multiple of the radix, then (1) the digits in which the squares of the numbers terminate are the same; (2) the sum

of this digit and the digit in which the product of the numbers terminates is equal to the radix.

Let the numbers be $ar+b$, $a'r-b$; r being the radix, and b the digit in the units' place in the first,

$$\text{their squares are } (a^2r+2ab)r+b^2, \text{ and } (a'^2r-2a'b)r+b^2;$$

and as these leave the same remainder when divided by r, the digits in which their squares terminate are the same.

Their product is $(aa'r+a'b-ab)r-b^2$; adding this to the square of either, we obtain a multiple of r, i. e. in the sum the digit in the units' place is 0, $\therefore$ the sum of the digits terminating the square of either, and the product of them, = the radix.

Ex. 62. A certain number when represented in the scale of 2 has each of its last three digits (counting from left to right) zero, and the next digit towards the left different from zero; when represented in either of the scales of 3 or 5, the last digit is zero, and the last but one different from zero; and in every other scale (twelve scales excepted) the last digit is different from zero. What is the number?

It appears that the number is divisible by 2^3 but not by 2^4; also by 3, but not by 3^2; and by 5, but not by 5^2;

$\therefore$ it may be expressed as $2^3.3.5.m$, where m does not contain 2, 3, or 5.

The last digit will be zero, whenever the number is expressed in a scale whose radix is any divisor of $2^3.3.5.m$.

The number of divisors of this number (including 1 and itself) is $(3+1)(1+1)(1+1)(p+1)(q+1)\ldots$, where p, q .. are the indices of the prime factors in m, *i. e.* $16(p+1)(q+1)\ldots$

But the question gives only 12 divisors besides the numbers 2, 3, 5;

$$\therefore\ 16(p+1)(q+1)\ldots \text{ is not} > 12+3+1;$$

$$\textit{i. e.} \text{ not} > 16;$$

$$\therefore\ p=0,\ q=0, \text{ \&c., and } \therefore\ m=1.$$

Consequently the number in question is $2^3.3.5=120$.

Ex. 63. Prove that every even power of every odd number when divided by 8 leaves 1 for a remainder.

Every such power is of the form $(2n+1)^{2m}$, i. e. $(4n^2+4n+1)^m$, or $\{4n(n+1)+1\}^m$; and as $n(n+1)$ is divisible by 2, this becomes of the form $(8p+1)^m$; and when this is expanded, all terms except the last involve powers of $8p$ multiplied by integral coefficients, and the last term $=1$; $\therefore$ when it is divided by 8, the remainder $=1$.

Ex. 64. If n be a prime number, prove that

$$1^n+2^n+3^n+\ldots\ldots+(rn)^n$$

is a multiple of n.

By similar reasoning to that used in Fermat's Theorem,
$(1+2+3+\ldots+rn)^n=1^n+2^n+3^n+\ldots+(rn)^n+nP$, where P is an integer,

$$i.\,e.\ \left\{\frac{rn(rn+1)}{2}\right\}^n-nP=1^n+2^n+3^n+\ldots+(rn)^n,$$

the left-hand member is evidently divisible by n,

$\therefore\ 1^n+2^n+3^n+\ldots+(rn)^n$ is divisible by n.

Ex. 65. If n be any prime number excepting 2, and N any odd number prime to n, prove that $N^{n-1}-1$ is divisible by $8n$.

By Fermat's Theorem, $N^{n-1}-1$ is divisible by n;

Also from Ex. 63 above, $N^{n-1}-1$ is divisible by 8, (for n must necessarily be odd);

$\therefore\ N^{n-1}-1$ being a common multiple of n and 8 is equal to, or a multiple of, their L. C. M., i.e. $8n$;

$\therefore\ N^{n-1}-1$ is divisible by $8n$.

Ex. 66. A gentleman being asked the size of his paddock replied:—Between one and two roods: also were it smaller by 3 square yards it would be a square number of square yards; and if my brother's paddock, which is a square number of square yards, were larger by one square yard, it would be exactly half as large as mine. What was the size of his paddock?

Let x^2+3 be the number of square yards in his paddock;

y^2 .... in his brother's.. ...

$$\therefore\ x^2+3=2(y^2+1);$$

$$\therefore\ x^2-2y^2=-1.$$

Now in the convergence to $\sqrt{2}$ the quotients are

1, 2, 2, 2, &c.,

and the number of recurring quotients is odd, being 1.

The convergents are

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \&c.;$$

$\therefore$ the solutions of the above equation are

$$x = 1,\ 7,\ 41,\ 239,\ \&c.;$$

$$y = 1,\ 5,\ 29,\ 169,\ \&c.$$

The question also gives $x^2+3 > 1210 < 2420$;

$$\therefore\ x = 41,$$

and the size of his paddock $= 41^2+3$,

$$= 1684 \text{ square yards.}$$

Ex. 67. *A* walks at a uniform speed, known to be greater than 3, and less than 4, miles an hour, between two places 20 miles apart. An hour having elapsed since *A*'s departure, *B* starts after him from the same place, walking at the uniform speed of 4 miles an hour. Shew that the odds are 2 to 1 against *B*'s overtaking *A*.

Let $3+x$ miles per hour be *A*'s rate of walking.

Then the time which elapses from *A*'s departure until his arrival at the end of his journey is $\dfrac{20}{3+x}$ hours.

The time before *B* arrives there is $1+\dfrac{20}{4}$, or 6, hours.

B will overtake *A*, or not, according as $6 \begin{smallmatrix}<\\>\end{smallmatrix} \dfrac{20}{3+x}$, *i. e.* according as $x \begin{smallmatrix}<\\>\end{smallmatrix} \dfrac{1}{3}$; but x is known to be between 0 and 1; therefore the odds are 2 : 1 against *B*'s overtaking *A*.

Ex. 68. When $2n$ dice are thrown, prove that the sum of the numbers turned up is more likely to be $7n$ than any other number; and when $2n+1$ dice are thrown, prove that the sum of the numbers turned up is more likely to be $7n+3$, or $7n+4$, than any other number, these being equally probable.

If m dice be thrown, the number of ways in which the sum of the numbers thrown is r

$$= \text{coefficient of } x^r \text{ in } (x+x^2+x^3+x^4+x^5+x^6)^m,$$

by Ex. 54, above,

$$= \text{coefficient of } x^{r-m} \text{ in } (1+x+x^2+x^3+x^4+x^5)^m;$$

therefore the most probable sum will be that value of r for which the coefficient of

$$x^{r-m} \text{ in } (1+x+x^2+x^3+x^4+x^5)^m \text{ is the greatest.}$$

If $m=2n$, the expansion has a middle term whose coefficient is greater than any other; and then $r-m=$ the corresponding index of $x=\frac{1}{2}.5m$, or $r=7n$.

If $m=2n+1$, there are two middle terms whose coefficients are equal to each other, and greater than any other coefficient; and then

$$r-m=\tfrac{1}{2}(5m-1), \text{ or } \tfrac{1}{2}(5m+1);$$

$$\therefore\ r=7n+3, \text{ or } 7n+4,$$

these being equally probable.

Ex. 69. A handful of shot is taken at random out of a bag: what is the chance that the number of shot in the handful is prime to the number of shot in the bag? Ex. Let the number of shot in the bag be 105.

Let the number of shot in the bag be $a^p b^q c^r \ldots$ where $a, b, c\ldots$ are prime numbers.

If the natural numbers taken in order be arranged in sets each containing a of them, the first $a-1$ in each are prime to the number a; and since the number of shot in the bag is a multiple of a, *i.e.* contains a certain number of such sets, the chance of the number drawn being prime to a is

$$\frac{a-1}{a} \text{ or } 1-\frac{1}{a}.$$

So the chance of its being prime to b is $1-\frac{1}{b}$, &c.: these probabilities being entirely independent of each other. If the number of shot drawn be prime to the number in the bag, it must be prime to $a, b, c\ldots$; therefore the probability required is

$$\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\ldots$$

Ex. If the number be 105, *i.e.* $3\times5\times7$, the chance is

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)=\frac{2}{3}.\frac{4}{5}.\frac{6}{7}=\frac{16}{35}.$$

Ex. 70. A coin is to be tossed twice: what is the chance that head will turn up at least once?

Point out the error in the following solution by D'Alembert:—Only three different events are possible; (1) head the first time, which makes it unnecessary to toss again; (2) tail the first time and head the second; (3) tail both times: of these three events two are favourable: therefore the required chance is $\frac{2}{3}$.

The chance of not turning up head in any throw is $\frac{1}{2}$; therefore the chance of not turning up head in the given case is $\frac{1}{4}$; therefore the required chance is $1-\frac{1}{4}=\frac{3}{4}$.

The error in D'Alembert's solution is, that each of the three events are considered equally probable, whereas (1) might be resolved into two, viz. head both times, head the first time and tail the second, thus giving 4 events equally probable, of which 3 are favourable. In fact the probability of (1) is $\frac{1}{2}$, but of (2) and (3) it is $\frac{1}{4}$.

Ex. 71. In each of n caskets are p jewels worth 1, 2, 3,...p guineas respectively: a person being allowed to draw a jewel from each casket, find (1) the most probable collective value of the jewels he will draw: (2) the value of his expectation. (3) What would be the value of his expectation, if he were allowed to draw r jewels from each casket? If $p=2$, and one jewel be drawn from each casket, what is the chance (4) that the value of the jewels drawn is $n+r$ guineas? (5) that the collective value of the jewels drawn is that collective value which is most probable?

(1) The number of ways in which the value of r guineas can be drawn

$$=\text{coefficient of } x^r \text{ in } (x+x^2+x^3+\ldots+x^p)^n,$$
$$=\text{coefficient of } x^{r-n} \text{ in } (1+x+x^2+\ldots+x^{p-1})^n.$$

If $n(p-1)$ be even, there is a middle term whose coefficient is the greatest.

If $n(p-1)$ be odd, there are 2 middle terms whose coefficients are equal to each other, and are greater than any of the others.

And the index of x, or $r-n$, in these cases

$$=\frac{1}{2}n(p-1), \text{ and } \frac{1}{2}n(p-1)\pm\frac{1}{2};$$

$$\therefore\ r=\frac{1}{2}n(p+1),$$

if $n(p-1)$ be even, *i. e.* if either n be even, or p odd,

$$\text{and } r=\frac{1}{2}n(p+1)\pm\frac{1}{2},$$

if n be odd and p even; these being equally probable.

And the most probable collective value drawn will be the number of guineas which can be drawn in the greatest number of ways, i.e. $\frac{n}{2}(p+1)$, if either n be even or p odd,

$$\text{and } \frac{n}{2}(p+1) \pm \frac{1}{2}, \text{ if } n \text{ be odd and } p \text{ even.}$$

(2) The value of his expectation from each casket is

$$\frac{1}{p}(1+2+3+\ldots+p) = \frac{p+1}{2} \text{ guineas};$$

therefore the value of his expectation in drawing from n caskets

$$= \frac{n(p+1)}{2} \text{ guineas.}$$

(3) If he were allowed to draw r jewels from each casket, the value of his expectation would become r times as great, and therefore would

$$= \frac{rn(p+1)}{2} \text{ guineas.}$$

(4) The number of ways in which $n+r$ guineas can be drawn $=$ coefficient of x^{n+r} in $(x+x^2)^n =$ coefficient of x^r in $(1+x)^n$.

The whole number of ways in which any drawing can take place

$$= \text{sum of the coefficients in } (x+x^2)^n = 2^n;$$

therefore chance of drawing $n+r$ guineas is

$$\frac{1}{2^n} \cdot \frac{\lfloor n}{\lfloor r \cdot \lfloor n-r}.$$

(5) The most probable collective value is, from (1),

$$\frac{3n}{2}, \text{ if } n \text{ be even, or } \frac{3n}{2} \pm \frac{1}{2}, \text{ if } n \text{ be odd.}$$

(i) Let n be even, $=2m$; then the chance of drawing $3m$ guineas

$$= \frac{1}{2^n} \cdot \frac{\lfloor n}{\lfloor m \cdot \lfloor m} = \frac{\lfloor n}{(2^m \cdot \lfloor m)^2} = \frac{\lfloor n}{(2.4.6\ldots n)^2}.$$

(ii) Let n be odd, $=2m+1$; then the chance of drawing $3m+1$ or $3m+2$ guineas

$$= \frac{1}{2^n} \cdot \frac{\lfloor n}{\lfloor m \cdot \lfloor m+1} = \frac{\lfloor n}{\{2.4.6\ldots 2m\}\{2.4.6\ldots(2m+2)\}},$$

$$= \frac{\lfloor n}{\{2.4.6\ldots(n-1)\}\{2.4.6\ldots(n+1)\}}.$$

Ex. 72. A person borrows £c on the following terms. It is to be paid off in n years; and at the end of each year is to be paid interest at a given rate r on the sum remaining unpaid at the beginning of the year, together with such a portion of the principal that the whole sum paid on account of principal and interest together shall be the same for every year. Investigate a formula for the sum to be paid every year.

Let x = each payment, and let $R = 1+r$.

At the beginning of the 2nd year, the sum left unpaid $\Big\} = cR - x$,

.... 3rd ...... $= cR^2 - xR - x$,

.... 4th ...... $= cR^3 - xR^2 - xR - x$,

$$= cR^3 - x(R^2 + R + 1),$$

$$= cR^3 - x . \frac{R^3-1}{R-1},$$

.... 5th ...... $= cR^4 - x . \frac{R^3-1}{R-1} . R - x$,

$$= cR^4 - x . \frac{R^4-1}{R-1} .$$

&c. = &c.

..............................$(n+1)^{\text{th}}$...... $= cR^n - x . \dfrac{R^n-1}{R-1}$:

and, by question, this = 0;

$$\therefore\ cR^n = x . \frac{R^n-1}{R-1};$$

$$\therefore\ x = cR^n . \frac{R-1}{R^n-1} = c . \frac{r(1+r)^n}{(1+r)^n-1} .$$

Ex. 73. From the gold fields are brought $2n$ specimens of gold dust, no two of which are of the same degree of fineness. Each specimen is divided into as many equal portions as is necessary to the following operation, viz. to form as many different mixtures as possible by taking a portion of dust from some specimen and mixing it with a portion from some other specimen. Each of these mixtures is now divided into as many equal portions as is necessary to the following operation, viz. to form as many new mixtures as possible by mixing together portions from any n of the former mixtures. Prove that $1 . 3 . 5 . 7 ...... (2n-1)$ of the latter mixtures will be of the same degree of fineness as a mixture formed by mixing together all the dust of $2n$ specimens exactly like the original specimens.

Call the specimens in order 1, 2, 3, 4, ...$2n$.

Let x_r = number of grains of gold in an ounce of specimen r.

a_r = number of ounces in one portion of specimen r*.

μ_r^ρ = number of grains of gold in a mixture of one portion of specimen r with one portion of specimen ρ.

First:—Let there be 4 specimens.

$$\text{Then}\quad \begin{aligned} a_1x_1+a_2x_2&=\mu_1^2,\\ a_1x_1+a_3x_3&=\mu_1^3,\\ a_1x_1+a_4x_4&=\mu_1^4,\\ a_2x_2+a_3x_3&=\mu_2^3,\\ a_2x_2+a_4x_4&=\mu_2^4,\\ a_3x_3+a_4x_4&=\mu_3^4,\end{aligned}$$

give the first mixtures: from which we get

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4=\mu_1^2+\mu_3^4=\mu_1^3+\mu_2^4=\mu_1^4+\mu_2^3.$$

Now each of the first mixtures is (by question) to be divided into five equal portions.

Hence three of the second mixtures will contain

$$\frac{\mu_1^2+\mu_3^4}{5},\quad \frac{\mu_1^3+\mu_2^4}{5},\quad \frac{\mu_1^4+\mu_2^3}{5},\ \text{or}\ \frac{a_1x_1+a_2x_2+a_3x_3+a_4x_4}{5}$$

grains of gold out of

$$\frac{a_1+a_2+a_3+a_4+a_5}{5}$$

ounces of dust, and will therefore be of the same degree of fineness as the mixture of the four original specimens.

Next let there be $2n-2$ specimens, viz. specimens 3, 4, 5, 6, ...$2n$.

And let P_{2n-2} represent the number of second mixtures which are of the same degree of fineness as a mixture composed of all $2n-2$ specimens. Then the introduction of two new specimens, viz. specimens 1, 2, will introduce the following first mixtures in addition to the first mixtures when the number of specimens is $2n-2$; viz.

$$\text{(i)}\quad \left.\begin{aligned} (1)\ldots a_1x_1+a_2x_2&=\mu_1^2\\ (2)\ldots a_1x_1+a_3x_3&=\mu_1^3\\ (3)\ldots a_1x_1+a_4x_4&=\mu_1^4\\ \ldots\ldots\ldots\ldots &=\ldots\\ (2n-1)\ldots a_1x_1+a_{2n}x_{2n}&=\mu_1^{2n}\end{aligned}\right\}\ 2n-1 \text{ in number,}$$

* These units are arbitrary.

together with

$$\text{(ii)} \qquad \left.\begin{array}{l} a_2x_2+\ a_3x_3\ \ =\mu_2^{\,3} \\ a_2x_2+\ a_4x_4\ \ =\mu_2^{\,4} \\ a_2x_2+\ a_5x_5\ \ =\mu_2^{\,5} \\ \dots\dots\dots\dots = \dots \\ a_2x_2+a_{2n}x_{2n}=\mu_2^{\,2n} \end{array}\right\} \ 2n-2 \text{ in number.}$$

Now when there are $2n-2$ specimens 3, 4, 5, 6...$2n$, each of the P_{2n-2} second mixtures contains the proportion of

$$a_3x_3+a_4x_4+\dots+a_{2n}x_{2n}$$

grains of gold to $a_3+a_4+\dots+a_{2n}$ ounces of dust: add to each of them the proper proportion of $\mu_1^{\,2}$, then from equation (1) we have P_{2n-2} second mixtures of $2n$ specimens, each containing the proportion of $a_1x_1+a_2x_2+a_3x_3+\dots+a_{2n}x_{2n}$ grains of gold to

$$a_1+a_2+a_3+\dots+a_{2n} \text{ ounces of dust.}$$

Again, equations (ii) shew that the mixtures containing specimen 2 (excepting that which contains specimen 3 in addition to specimen 2) may replace the mixtures containing specimen 3 in forming the second mixtures of $2n-2$ specimens; this gives P_{2n-2} second mixtures containing the proportion of

$$a_2x_2+a_4x_4+\dots+a_{2n}x_{2n}$$

grains of gold to $a_2+a_4+\dots+a_{2n}$ ounces of dust: add to each of them the proper proportion of $\mu_1^{\,3}$; then from equation (2) we have P_{2n-2} second mixtures of $2n$ specimens, each containing the proportion of $a_1x_1+a_2x_2+\dots+a_{2n}x_{2n}$ grains of gold to

$$a_1+a_2+a_3+\dots+a_{2n} \text{ ounces of dust.}$$

Similarly each of the $2n-1$ equations gives P_{2n-2} of such second mixtures.

Hence there are altogether $(2n-1)P_{2n-1}$ second mixtures, each of which is of the same degree of fineness with a mixture of all the dust in the $2n$ original specimens.

$$\begin{aligned} \text{Hence}\quad P_{2n} &= (2n-1)P_{2n-2}, \\ &= (2n-1)(2n-3)P_{2n-4}, \\ &= \dots\dots\dots\dots\dots \\ &= (2n-1)(2n-3)\dots 5P_4, \\ &= (2n-1)(2n-3)\dots 5.3.1\,; \because P_4=3. \end{aligned}$$

$$4x^2-4ax+a^2=3a^2,$$

$$2x=(1-\sqrt{3})a \dots\dots\dots (\alpha),$$

$$\therefore\ x=\frac{1}{2}(1-\sqrt{3})a.$$

Also $x^2-y^2=\frac{(a-2x)^3}{27a}=\frac{3\sqrt{3}.a^3}{27a}$, from (α), $=\frac{a^2}{3\sqrt{3}}$,

$$\therefore\ y^2=\frac{4-2\sqrt{3}}{4}.a^2-\frac{a^2}{3\sqrt{3}}=(1-\frac{11}{18}\sqrt{3})a^2,$$

$$\therefore\ y=\sqrt{\left(1-\frac{11}{18}\sqrt{3}\right)}a.$$

5. A railway train travels from A to C passing through B where it stops 7 minutes; two minutes after leaving B it meets an express train which started from C when the former was 28 miles on the other side of B: the express travels at double the rate of the other, and performs the journey from C to B in $1\frac{1}{2}$ hours; and if on reaching A it returned at once to C it would arrive 3 minutes after the first train. Find the distances between A, B and C, and the speed of each train.

A B E C

Let x = speed, in miles *per hour*, of the ordinary train,

$\therefore\ 2x=$ .... express .....

$$\therefore\ BC=2x\times 1\tfrac{1}{2}=3x.$$

Let E be the place where the trains meet; then $BE=\frac{x}{30}$,

$$\text{and } \frac{28}{x}+\frac{9}{60}=\frac{CE}{2x}=\frac{BC-BE}{2x}=\frac{3}{2}-\frac{1}{60},$$

$$\therefore\ \frac{28}{x}=\frac{3}{2}-\frac{10}{60}=\frac{80}{60}=\frac{4}{3},$$

$$\therefore\ x=21,\ 2x=42,\ \text{and } BC=63.$$

Also, by the question, $\frac{EC}{x}+\frac{3}{60}=\frac{AE+AC}{2x}$,

$$=\frac{2AC-EC}{2x}=\frac{AC}{x}-\frac{EC}{2x},$$

$$\frac{AC}{x}=\frac{3EC}{2x}+\frac{1}{20}=\frac{3}{2}\left(3-\frac{1}{30}\right)+\frac{1}{20}=\frac{9}{2},$$

$$\therefore\ AC = \frac{9}{2}x = \frac{9\times 21}{2} = 94\tfrac{1}{2},$$

$$\text{and } AB = AC - BC = 94\tfrac{1}{2} - 63 = 31\tfrac{1}{2}.$$

6. To meet a deficiency of m millions in the revenue of a country, an *additional* tax of a per cent. was laid upon articles exported, and the tax upon imports was diminished c per cent.: in consequence of these alterations the value of the imports was increased so as to be n times as great as the exports, and the deficiency was made up. It was afterwards found that if the additional tax upon the exports had been a' per cent., and the tax upon imports diminished c' per cent., the values of the articles being altered as before, the deficiency would not have been made up by m' millions. Find the values of the exports and imports after the alteration of the tax.

Let x be the value of the exports *in cents.* after the alteration,

$\therefore\ nx =$ .... imports ....

$\therefore\ ax =$ the *gain* in exports ....

$ncx =$ ... *loss* ... imports ....

$$\therefore\ ax - ncx = m, \text{ by the question.}$$

$$\text{Also } a'x - nc'x = m - m', \ldots\ldots\ldots\ldots$$

$$\therefore\ \{a - a' + n(c' - c)\}x = m',$$

$$\therefore\ x = \frac{m'}{a - a' + n(c' - c)},$$

$$\text{and } nx = \frac{m'n}{a - a' + n(c' - c)}.$$

7. Fifty thousand voters, who have to return a member to an assembly, are divided into sections of equal size, and each section chooses an elector, the member being returned by the majority of such electors. There are two candidates, A and B. In those sections which return electors favourable to A, the majority is double the minority, while in those favourable to B, the minority forms only a tenth of the whole. After the primary elections a third candidate C comes forward, and is joined by so many electors of each party, that he is returned by a majority of 3 over A, and 14 over B. If C had not come forward, A would have been returned by a majority 19 less than the whole number of votes actually polled by C, and if the elections had been by the 50,000 voters *directly* between A and B, B would have had a majority of 6000. Find the number of sections.

Let x = the number of sections = number of *electors*;

$$\therefore \frac{50,000}{x} = \text{N}^\circ.\text{ of voters in each section;}$$

let y = N°. of sections favourable to A,

then $x-y$ = ... B;

$\therefore y \times \dfrac{50,000}{x}$ = N°. of *voters* in those sections which are favourable to A,

$$\therefore \text{by the question, } \left.\begin{aligned} y\times\frac{100,000}{3x} &= \text{votes for } A,\\ y\times\frac{50,000}{3x} &= \text{votes for } B,\end{aligned}\right\}\text{ in these sections.}$$

Again $(x-y).\dfrac{50,000}{x}$ = N°. of *voters* in the sections which are favourable to B,

$$\left.\begin{aligned} \text{and } (x-y).\frac{45,000}{x} &= \text{votes for } B,\\ (x-y).\frac{5,000}{x} &= \text{votes for } A,\end{aligned}\right\}\text{ in these sections.}$$

Now let z = the N°. of *electors* who voted for C, then

$$z+(z-3)+(z-14),\text{ or } 3z-17=x\ldots\ldots(1).$$

$$\text{Also } y-(x-y)=z-19,\text{ or } x-2y+z=19\ldots\ldots(2).$$

$$\text{And } y.\frac{100,000}{3x}+(x-y).\frac{5,000}{x}=y.\frac{50,000}{3x}+(x-y).\frac{45,000}{x}-6,000,$$

$$\text{or } \frac{50y}{3}=40(x-y)-6x\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (3), $y=\dfrac{3}{5}x$; from (1), $z=\dfrac{x}{3}+5\frac{2}{3}$,

$$\therefore \text{ from (2), } x=2y-z+19=\frac{6x}{5}-\frac{x}{3}-5\tfrac{2}{3}+19,$$

$$\therefore x=100.$$

II.

1. $2b\{\sqrt{x+a}-b\}+2c\{\sqrt{x-a}+c\}=a$; find x.

$$2a-4b\sqrt{x+a}+4b^2=4c\sqrt{x-a}+4c^2,$$

$$x+a-4b\sqrt{x+a}+4b^2=x-a+4c\sqrt{x-a}+4c^2,$$

$$\sqrt{x+a}-2b=\pm(\sqrt{x-a}+2c),$$

$$\sqrt{x+a}\mp\sqrt{x-a}=2(b\pm c),$$

$$x\mp\sqrt{x^2-a^2}=2(b\pm c)^2,$$

$$x^2-4(b\pm c)^2x+4(b\pm c)^4=x^2-a^2,$$

$$\therefore\ x=(b\pm c)^2+\frac{1}{4}\cdot\frac{a^2}{(b\pm c)^2}.$$

2. $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$; find x.

$$\sqrt{2x-1}-\sqrt{5x-4}=\sqrt{4x-3}-\sqrt{3x-2},$$

$$7x-5-2\sqrt{10x^2-13x+4}=7x-5-2\sqrt{12x^2-17x+6},$$

$$10x^2-13x+4=12x^2-17x+6,$$

$$2x^2-4x+2=0,$$

$$x^2-2x+1=0,$$

$$(x-1)^2=0,$$

$$\therefore\ x=1.$$

3. $8x^{\frac{3}{4}}+81=18x^{\frac{1}{2}}+45x^{\frac{1}{4}}$; find x.

For x write y^4, then $8y^3+81=18y^2+45y$,

$$\left(\frac{2y}{3}\right)^3+3=\frac{2}{3}y^2+\frac{5}{3}y,$$

$$\left(\frac{2y}{3}\right)^3+\left(\frac{3}{2}\right)^3=\frac{2}{3}y^2+\frac{5}{3}y+\frac{3}{8},$$

$$=\left(\frac{2}{3}y+\frac{3}{2}\right)\left(y+\frac{1}{4}\right),$$

$$\therefore\ \frac{2}{3}y+\frac{3}{2}=0,$$

$$y=-\frac{9}{4},$$

$$x=\left(-\frac{9}{4}\right)^4.$$

and $\left(\frac{2y}{3}\right)^2-\frac{2y}{3}\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2=y+\frac{1}{4},$

$$\left(\frac{2y}{3}\right)^2-3\cdot\frac{2y}{3}+\left(\frac{3}{2}\right)^2=\frac{1}{4},$$

$$\frac{2y}{3}=\frac{3}{2}\pm\frac{1}{2},\quad \therefore\ y=3,\ \text{or}\ \frac{3}{2},$$

and $x=y^4=3^4$, or $\left(\frac{3}{2}\right)^4$.

4. $$\left.\begin{array}{l}\dfrac{y-x+\sqrt{2xy-3x^2}}{y-2x}=3\cdot\dfrac{(2y-3x)^{\frac{3}{2}}+x^{\frac{3}{2}}}{(2y-3x)^{\frac{3}{2}}-x^{\frac{3}{2}}}\ldots(1),\\ \dfrac{y}{x^2}+\dfrac{16}{81}\left(x-\sqrt{x}-\dfrac{3}{4}\right)=\dfrac{4}{9x}(2\sqrt{xy}-\sqrt{y})\ldots(2);\end{array}\right\}\ \text{find } x \text{ and } y.$$

For $2y-3x$ write z, then, from (1),

$$3\cdot\frac{z^{\frac{3}{2}}+x^{\frac{3}{2}}}{z^{\frac{3}{2}}-x^{\frac{3}{2}}}=\frac{2y-3x+2\sqrt{x}\cdot\sqrt{2y-3x}+x}{(2y-3x)-x},$$

$$=\frac{\{(2y-3x)^{\frac{1}{2}}+x^{\frac{1}{2}}\}^2}{(2y-3x)-x}=\frac{(2y-3x)^{\frac{1}{2}}+x^{\frac{1}{2}}}{(2y-3x)^{\frac{1}{2}}-x^{\frac{1}{2}}},$$

$$=\frac{z^{\frac{1}{2}}+x^{\frac{1}{2}}}{z^{\frac{1}{2}}-x^{\frac{1}{2}}},$$

$$\therefore\ 3\cdot\frac{z^{\frac{3}{2}}+x^{\frac{3}{2}}}{z^{\frac{1}{2}}+x^{\frac{1}{2}}}=\frac{z^{\frac{3}{2}}-x^{\frac{3}{2}}}{z^{\frac{1}{2}}-x^{\frac{1}{2}}},$$

$$3z-3z^{\frac{1}{2}}x^{\frac{1}{2}}+3x=z-z^{\frac{1}{2}}x^{\frac{1}{2}}+x,$$

$$z-2z^{\frac{1}{2}}x^{\frac{1}{2}}+x=0,\quad \therefore\ z=x,\ \text{and}\ y=2x.$$

From (2), $\dfrac{y}{x^2}-\dfrac{4}{9}(2\sqrt{x}-1)\dfrac{\sqrt{y}}{x}+\dfrac{4}{81}(2\sqrt{x}-1)^2=\dfrac{16}{81},$

$$\frac{\sqrt{y}}{x}=\frac{2}{9}(2\sqrt{x}-1\pm2),$$

$$=\frac{2}{9}(2\sqrt{x}+1),\ \text{or}\ \frac{2}{9}(2\sqrt{x}-3)\ \ldots\ldots(\alpha).$$

Taking the 2nd value, and substituting $2x$ for y,

$$\frac{4}{9}x-\frac{2}{3}\sqrt{x}+\frac{1}{4}=\sqrt{2}+\frac{1}{4},$$

$$\frac{2}{3}\sqrt{x}=\frac{1}{2}(1\pm\sqrt{4\sqrt{2}+1}),$$

$$\therefore\ x=\frac{9}{16}(1\pm\sqrt{4\sqrt{2}+1})^2.$$

$$\text{And } y=2x=\frac{9}{8}(1\pm\sqrt{4\sqrt{2}+1})^2.$$

Other values of x and y will be found by taking the other value of $\frac{\sqrt{y}}{x}$ in (a).

5. The distance between the two termini A and Z of a railway is 100 miles. A train starting from A runs up-hill during the first 30 miles of its journey, the next 50 miles are on a level, and the remaining 20 are up-hill. The train may be supposed to travel 5 miles an hour faster on the horizontal road than when it is ascending a hill. There are to be stoppages at stations B, C, D, and E, at distances 20, $42\frac{1}{2}$, $67\frac{1}{2}$, and 90, miles respectively from A, and each stoppage may be supposed to cause a detention of 3 minutes. Find the time of arrival at B, C, D, and E, of the train which starts from A at $8^h.\ 0^m.$ and arrives at Z at $12^h.\ 42^m.$

Let $x=$ N°. of miles per hour *up-hill.*

$\therefore\ x+5=$ .... on the level,

and $\frac{50}{x+5}+\frac{50}{x}=$ whole time in hours for the whole journey;

but the whole time $=12^h.\ 42^m-8^h.\ 0^m-4\times3^m=4\frac{1}{2}^h$,

$$\therefore\ \frac{50}{x+5}+\frac{50}{x}=\frac{9}{2},$$

from which we get $x=20$, the No. of miles per hour up-hill.

$\therefore$ train arrives at B in one hour, that is, at 9^h, and leaves B at $9^h.\ 3^m.$

Between B and C ($22\frac{1}{2}$ miles) there are 10 miles up-hill, and $12\frac{1}{2}$ miles on the level, *each* of which occupies half an hour,

$\therefore$ train arrives at C in one hour after leaving A, that is, at $10^h.\ 3^m$, and leaves C at $10^h.\ 6^m.$

Between C and D the whole 25 miles are on the level, and will occupy 1 hour; $\therefore$ train arrives at D $11^h.\ 6^m$, and leaves D at $11^h.\ 9^m.$

Between D and E ($22\frac{1}{2}$ miles), $12\frac{1}{2}$ are on the level, and 10 up-hill, *each* of which occupies half an hour,

$\therefore$ train arrives at E at $12^h.\ 9^m.$

6. A number of vessels A_1, A_2, A_3,......A_r, A_{r+1}...A_m are arranged in a row. A_1 contains a quantity of wine, A_2 a quantity of water, and the remaining vessels A_3,...A_r, A_{r+1},...A_m, contain any quantity of any other fluids. $\frac{1}{n}$ of the wine in A_1 is taken from A_1 and added to the contents of A_2, $\frac{1}{n}$ of the mixture is taken from A_2 and poured into A_3, $\frac{1}{n}$ of the contents of A_3 is poured into A_4, and so on to the end of the series of vessels. Again, $\frac{1}{n}$ of wine remaining in A_1 is poured into A_2, $\frac{1}{n}$ of the contents of A_2 into A_3, and so on. A_1 is supposed never to receive any addition. It is found that 60 times the quantity of wine in the vessel A_r after $r-1$ abstractions of fluid from that vessel $=31$ times the quantity of wine in the same vessel after r abstractions. Also 59 times the quantity of water in A_r after $r-1$ abstractions of fluid from that vessel $=31$ times the quantity of water in the same vessel after r abstractions. Find the numerical values of r and n.

Let x=quantity of wine in A_1, then for 1st abstraction we have

$$\left.\begin{array}{l|l|l}
\text{in } A_1 \;\; x \text{ at 1st,} & \frac{x}{n} \text{ taken out,} & x\left(1-\frac{1}{n}\right) \text{ left in,} \\
\ldots A_2 \;\; \frac{x}{n} \;\ldots\ldots & \frac{x}{n^2} \;\ldots\ldots\ldots\ldots & \frac{x}{n}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots \\
\ldots A_3 \;\; \frac{x}{n^2} \;\ldots\ldots & \frac{x}{n^3} \;\ldots\ldots\ldots\ldots & \frac{x}{n^2}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots A_r \;\; \frac{x}{n^{r-1}} \;\ldots\ldots & \frac{x}{n^r} \;\ldots\ldots\ldots\ldots & \frac{x}{n^{r-1}}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots
\end{array}\right\}$$

for 2nd abstraction,

$$\left.\begin{array}{l|l|l}
\text{in } A_1 \;\; x\left(1-\frac{1}{n}\right) \text{ at 1st,} & \frac{x}{n}\left(1-\frac{1}{n}\right) \text{ taken out,} & x\left(1-\frac{1}{n}\right)^2 \text{ left in,} \\
\ldots A_2 \;\; \frac{2x}{n}\left(1-\frac{1}{n}\right) \;\ldots\ldots & \frac{2x}{n^2}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots\ldots & \frac{2x}{n}\left(1-\frac{1}{n}\right)^2 \;\ldots\ldots\ldots \\
\ldots A_3 \;\; \frac{3x}{n^2}\left(1-\frac{1}{n}\right) \;\ldots\ldots & \frac{3x}{n^3}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots\ldots & \frac{3x}{n^2}\left(1-\frac{1}{n}\right)^2 \;\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots A_r \;\; \frac{rx}{n^{r-1}}\left(1-\frac{1}{n}\right) \ldots\ldots & \frac{rx}{n^r}\left(1-\frac{1}{n}\right) \;\ldots\ldots\ldots\ldots & \frac{rx}{n^{r-1}}\left(1-\frac{1}{n}\right)^2 \ldots\ldots\ldots
\end{array}\right\}$$

for 3rd abstraction,

$$\left.\begin{array}{l|l|l}
\text{in } A_1 \;\; x\left(1-\dfrac{1}{n}\right)^2 \text{ at 1st,} & \dfrac{x}{n}\left(1-\dfrac{1}{n}\right)^2 \text{taken out,} & x\left(1-\dfrac{1}{n}\right)^3 \text{ left in,} \\
\ldots A_2 \;\; \dfrac{3x}{n}\left(1-\dfrac{1}{n}\right)^2 \;\ldots\ldots\ldots & \dfrac{3x}{n^2}\left(1-\dfrac{1}{n}\right)^2 \ldots\ldots\ldots\ldots & \dfrac{3x}{n}\left(1-\dfrac{1}{n}\right)^3 \;\ldots\ldots\ldots \\
\ldots A_3 \;\; \dfrac{6x}{n^2}\left(1-\dfrac{1}{n}\right)^2 \;\ldots\ldots\ldots & \dfrac{6x}{n^3}\left(1-\dfrac{1}{n}\right)^2 \ldots\ldots\ldots\ldots & \dfrac{6x}{n^2}\left(1-\dfrac{1}{n}\right)^3 \;\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots A_r \;\; \dfrac{r(r+1)}{1\,.\,2}\cdot\dfrac{x}{n^{r-1}}\left(1-\dfrac{1}{n}\right)^2\ldots & \dfrac{r(r+1)}{1\,.\,2}\cdot\dfrac{x}{n^r}\left(1-\dfrac{1}{n}\right)^2 \;\ldots & \dfrac{r(r+1)}{1\,.\,2}\cdot\dfrac{x}{n^{r-1}}\left(1-\dfrac{1}{n}\right)^3\ldots
\end{array}\right\}$$

and so on to the r^{th} abstraction. Hence the quantity of wine left in A_r after r abstractions is

$$\frac{r(r+1)(r+2)\ldots(r+\overline{r-2})}{1\,.\,2\,.\,3\;\;\ldots\;\;(r-1)}\cdot\frac{x}{n^{r-1}}\left(1-\frac{1}{n}\right)^r = w_r \text{ suppose,}$$

and after $\overline{r-1}$ abstractions is

$$\frac{r(r+1)(r+2)\ldots(r+\overline{r-3})}{1\,.\,2\,.\,3\;\;\ldots\;\;(r-2)}\cdot\frac{x}{n^{r-1}}\left(1-\frac{1}{n}\right)^{r-1} = w_{r-1}\ldots\ldots$$

$\therefore$ by the question, $\dfrac{60}{31}\cdot\dfrac{w_{r-1}}{w_r}=1,$

$$\text{or } \frac{60}{31}\cdot\frac{r-1}{2r-2}\cdot\left(1-\frac{1}{n}\right)^{-1}=1,$$

$$\text{or } \frac{60}{31}\cdot\frac{1}{2}\cdot\frac{n}{n-1}=1,\quad \frac{n}{n-1}=\frac{31}{30},\quad \therefore\; n=31.$$

Similarly, observing that the *water* will have to come through one vessel less than the wine, we have the *water* in A_r

$$\text{after } r \text{ abstractions} = \frac{(r-1)r(r+1)\ldots(2r-3)}{1\,.\,2\,.\,3\;\;\ldots\;\;(r-1)}\cdot\frac{y}{n^{r-2}}\left(1-\frac{1}{n}\right)^r,$$

$$\text{after } \overline{r-1} \text{ abstractions} = \frac{(r-1)r(r+1)\ldots(2r-4)}{1\,.\,2\,.\,3\;\;\ldots\;\;(r-2)}\cdot\frac{y}{n^{r-2}}\left(1-\frac{1}{n}\right)^{r-1},$$

$\therefore$ by the question, $\dfrac{59}{31}\cdot\left(1-\dfrac{1}{n}\right)^{-1}\cdot\dfrac{r-1}{2r-3}=1,$

$$\frac{59}{31}\cdot\frac{31}{30}=\frac{2r-3}{r-1},\quad \text{or } 59r-59=60r-90.$$

$$\therefore\; r=31.$$

7. Two points P and Q are connected by a wire (A), $\frac{1}{8}$th of an inch in diameter and 50 miles in length, which is used for transmitting a galvanic current. It is required to replace the wire (A) by three others (a), (b), and (c), composed of different metals, and of lengths 50, 60 and 70 miles respectively. These new wires must be of such diameters that the current, which previously passed along (A) may be divided so that the quantities which pass along (a), (b), and (c) may be as 3, 4, and 5. The quantity of galvanic fluid that will pass along a wire is supposed to vary inversely as the resistance, and the resistance to vary directly as the length of wire to be traversed, inversely as the sectional area of the wire, and inversely as the conductibility. Also the sectional area of a wire varies as the square of its diameter. It is found by experiment that the quantity of galvanic fluid which will pass along a portion of the wire (A), $\frac{1}{8}$th of an inch in diameter and 15 yds. long, may be denoted by 1000 k. Also portions of wire $\frac{1}{10}$th of an inch in diameter, composed of the same metals as (a), (b), and (c), of lengths 20, 10, and 40 yds. respectively, are capable of transmitting quantities of galvanic fluid 750 k, 5400 k, and 3500 k respectively. Find the least possible diameters of wires (a), (b), and (c), in order that the above conditions may be satisfied.

Let x, y, z, be the required diameters of the wires (a), (b), (c), respectively in inches,

Q the quantity of galvanic fluid passing from P to Q along (A), q_a, q_b, q_c, quantities.. (a), (b), and (c), respectively.

K the conductibility of the metal of which (A) is composed, K_a, K_b, K_c, ...(a), (b), (c)......, respectively.

Now quantity of galvanic fluid passing along a wire

$$\propto \frac{1}{\text{resistance}} \propto \frac{1}{\dfrac{\text{length}}{\text{sect}^l.\text{ area}} \cdot \dfrac{1}{\text{cond}^y.}}$$

$$\propto \frac{(\text{sect}^l.\text{ area}) \times \text{cond}^y.}{\text{length}} \propto \frac{(\text{diam}^r.)^2 \times \text{cond}^y.}{\text{length}};$$

$$\therefore \frac{Q}{1000k} = \frac{15}{50 \times 1760}, \text{ or } Q = \frac{30}{176}.k.$$

$$\text{Also } \frac{q_a}{750k} = \frac{x^2 \times K_a}{50 \times 1760} \div \frac{\left(\frac{1}{10}\right)^2.K_a}{20}, \text{ or } q_a = \frac{3000}{176}.kx^2.$$

$$\frac{q_b}{5400k} = \frac{y^2 \times K_b}{60 \times 1760} \div \frac{\left(\frac{1}{10}\right)^2 . K_b}{10}, \text{ or } q_b = \frac{9000}{176} . ky^2.$$

$$\frac{q_c}{3500k} = \frac{z^2 \times K_c}{70 \times 1760} \div \frac{\left(\frac{1}{10}\right)^2 . K_c}{40}, \text{ or } q_c = \frac{20000}{176} . kz^2.$$

Now $q_a : q_b : q_c = 3 : 4 : 5 \ldots (1),$ and $q_a + q_b + q_c = Q \ldots\ldots\ldots (2),$ } by the question,

$\therefore$ from (1) $\quad q_a+q_b+q_c : q_a :: 12 : 3 :: 4 : 1,$

$q_a+q_b+q_c : q_b :: 12 : 4 :: 3 : 1,$

$q_a+q_b+q_c : q_c :: 12 : 5.$

$\therefore$ by (2), $\quad q_a = \frac{1}{4}Q, \quad q_b = \frac{1}{3}Q, \quad q_c = \frac{5}{12}Q.$

$$\therefore \; 3000 . \frac{kx^2}{176} = \frac{1}{4} . 30 . \frac{k}{176}, \quad x^2 = \frac{1}{400}, \quad \therefore \; x = \frac{1}{20}.$$

$$9000 . \frac{ky^2}{176} = \frac{1}{3} . 30 . \frac{k}{176}, \quad y^2 = \frac{1}{900}, \quad \therefore \; y = \frac{1}{30}.$$

$$20000 . \frac{kz^2}{176} = \frac{5}{12} . 30 . \frac{k}{176}, \quad z^2 = \frac{1}{1600}, \quad \therefore \; z = \frac{1}{40}.$$

III.

1. $\quad (x+a)\left(1+\frac{1}{x^2+a^2}\right)+\sqrt{2ax}\left(1-\frac{1}{x^2+a^2}\right)=2$; find x.

$$x+a+\sqrt{2ax}+\frac{x+a-\sqrt{2ax}}{(x+a-\sqrt{2ax})(x+a+\sqrt{2ax})}=2,$$

$$x+a+\sqrt{2ax}+\frac{1}{x+a+\sqrt{2ax}}=2,$$

$$(x+a+\sqrt{2ax})^2-2(x+a+\sqrt{2ax})+1=0,$$

$$x+a+\sqrt{2ax}=1,$$

$$x^2+2ax+a^2-2x-2a+1=2ax,$$

$$x^2-2x+1=2a-a^2,$$

$$\therefore \; x=1\pm\sqrt{2a-a^2}.$$

2. $\dfrac{1}{6x^2-7x+2}+\dfrac{1}{12x^2-17x+6}=8x^2-6x+1$; find x.

$$\frac{1}{(2x-1)(3x-2)}+\frac{1}{(3x-2)(4x-3)}=(2x-1)(4x-1),$$

$$\frac{6x-4}{(2x-1)(3x-2)(4x-3)}=(2x-1)(4x-1),$$

$\therefore\ 3x-2=0,$ $\therefore\ x=\dfrac{2}{3},$ | and $2=(2x-1)^2(4x-1)(4x+3),$ $=y^2(2y-1)(2y+1)$, writing y for $2x-1$, $=4y^4-y^2,$

$$\therefore\ 4y^4-y^2+\frac{1}{16}=2+\frac{1}{16}=\frac{33}{16},$$

$$\therefore\ 2y^2=\frac{1}{4}(1\pm\sqrt{33}),\ \text{and}\ y=\pm\sqrt{\frac{1}{8}(1\pm\sqrt{33})},$$

$$\therefore\ x=\frac{1}{2}\left\{1\pm\sqrt{\frac{1}{8}(1\pm\sqrt{33})}\right\}.$$

3. $$\left.\begin{aligned}(x^2+y^2+c^2)^{\frac{1}{2}}+(x-y+c)^{\frac{2}{3}}&=2(4xy)^{\frac{1}{3}}\ldots(1),\\ \frac{1}{y}=\frac{1}{x}+\frac{1}{c}&\ldots\ldots\ldots\ldots(2);\end{aligned}\right\}\ \text{find } x \text{ and } y.$$

Since $(x-y+c)^2=x^2+y^2+c^2+2xyc\left(\dfrac{1}{y}-\dfrac{1}{x}-\dfrac{1}{c}\right),$

and from (2), $\dfrac{1}{y}-\dfrac{1}{x}-\dfrac{1}{c}=0,$

$$\therefore\ x^2+y^2+c^2=(x-y+c)^2,$$

and from (1), $(x-y+c)^2=4xy,$

but from (2), $xy=c(x-y),$

$$\therefore\ (x-y+c)^2=4c(x-y),$$

$$\therefore\ (x-y-c)^2=0,\quad \therefore\ y=x-c,$$

from (2), $y=\dfrac{cx}{x+c},\quad \therefore\ x^2-c^2=cx,$

$$x^2-cx+\frac{c^2}{4}=\frac{5c^2}{4},$$

$$\therefore\ x=\frac{1}{2}(1\pm\sqrt{5}).c,$$

and $y=\dfrac{1}{2}(-1\pm\sqrt{5}).c.$

4. $$\left.\begin{aligned} 2(x^2+xy+y^2-a^2)+\sqrt{3}(x^2-y^2)&=0\ldots(1),\\ 2(x^2-xz+z^2-b^2)+\sqrt{3}(x^2-z^2)&=0\ldots(2),\\ y^3-c^3+3(yz^2-c^3)&=0\ldots(3);\end{aligned}\right\}\ \text{find } x \text{ and } y.$$

Multiply (1) by 2, $3(x+y)^2+(x-y)^2+2\sqrt{3}(x^2-y^2)=4a^2$,

$$\therefore\ \sqrt{3}(x+y)+x-y=2a.$$

Similarly from (2), $\sqrt{3}(x-z)+x+z=2b$,

$\therefore$ subtracting, $\sqrt{3}(y+z)-(y+z)=2(a-b)$,

$$\therefore\ y+z=\frac{2}{\sqrt{3}-1}(a-b)=(\sqrt{3}+1)(a-b)=m,\ \text{suppose.}$$

Again from (3), $2y^3+6yz^2=8c^3$,

or $(y+z)^3+(y-z)^3=8c^3$,

$$\therefore\ y-z=\sqrt[3]{8c^3-m^3},$$

$$\therefore\ y=\frac{1}{2}(m+\sqrt[3]{8c^3-m^3}),$$

$$\text{and } z=\frac{1}{2}(m-\sqrt[3]{8c^3-m^3}).$$

Also $(\sqrt{3}+1)x=2b+(\sqrt{3}-1)z=a+b-\frac{1}{2}(\sqrt{3}-1)\sqrt[3]{8c^3-m^3}$,

$$\therefore\ x=\frac{1}{\sqrt{3}+1}\left\{a+b-\frac{1}{2}(\sqrt{3}-1)\sqrt[3]{8c^3-m^3}\right\}.$$

5. *A* lends one half of his money to *B* at 5 per cent. per annum simple interest, and the remaining half he invests in the three per cents. at 90. *B* pays the interest regularly during the first five years, but afterwards neglects to do so till other five years' interest is due, when *A* calls in all his money, and *B* becomes a bankrupt paying 10*s*. in the pound. *A* sells out when the funds are at 81, and then he finds that the whole sum he has received as principal and interest in the ten years exceeds the sum that he originally possessed by £34. 13*s*. 4*d*. How much did he lend *B*.

Let x£ = the sum lent to B = sum invested in the 3 per cents.,

$\therefore$ interest received from the funds in 10 years for every 90£ is 30£, that is, for x is $\frac{x}{3}$; and the principal at last is sold for $\frac{81}{90}x$, or $\frac{9x}{10}$.

$\therefore$ whole sum received on this account $= \dfrac{x}{3} + \dfrac{9x}{10}$, or $\dfrac{37x}{30}$.

B pays in interest $5 . \dfrac{x}{20}$, or $\dfrac{x}{4}$. At the end of the 10 years he owes A, in principal and interest, $x + \dfrac{x}{4}$, or $\dfrac{5x}{4}$; and he pays only $\dfrac{5x}{8}$ for this; $\therefore$ whole sum received on this account $= \dfrac{x}{4} + \dfrac{5x}{8}$, or $\dfrac{7x}{8}$.

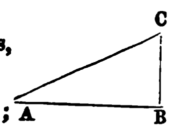
$\therefore$ by the question, $\dfrac{37x}{30} + \dfrac{7x}{8} = 2x + 34\frac{2}{3}$,

$$148x + 105x = 240x + 4160,$$

$$13x = 4160,$$

$$\therefore x = 320£.$$

6. A, B, and C, are three villages. The road from A to B is level, and C is on a hill above A and B. The distances AB, BC, and CA, are respectively 24, 14·4, and 28·8, miles. P walks up hill $\frac{1}{4}$th slower, and down hill $\frac{1}{3}$rd faster, than when the road is level. Q walks up hill $\frac{1}{5}$th slower, and down hill $\frac{1}{5}$th faster, than when the road is level. P travels round in the direction ACB in 30 minutes less than Q requires to go round in the opposite direction. Q travels round in the direction ACB in 1 hour 48 minutes more than P takes to make the circuit in the opposite direction; find the rates of each on a level road. Also supposing them both to start together from A in opposite directions, find their points of meeting.



I. Let $12x = P$'s rate per hour on the level, in miles,
then $9x = \ldots\ldots\ldots\ldots\ldots\ldots$ up-hill, $\ldots\ldots\ldots$
and $16x = \ldots\ldots\ldots\ldots\ldots\ldots$ down-hill$\ldots\ldots\ldots$;
let $5y = Q$'s$\ldots\ldots\ldots\ldots$ on the level, $\ldots\ldots\ldots$ | $AB = 24$ miles,
then $4y = \ldots\ldots\ldots\ldots\ldots\ldots$ up-hill, $\ldots\ldots\ldots$ | $BC = 14{\cdot}4\ldots\ldots$
and $6y = \ldots\ldots\ldots\ldots\ldots\ldots$ down-hill$\ldots\ldots\ldots$ | $AC = 28{\cdot}8\ldots\ldots$

$$\therefore \text{ by the question, } \frac{28{\cdot}8}{9x} + \frac{14{\cdot}4}{16x} + \frac{24}{12x} = \frac{24}{5y} + \frac{14{\cdot}4}{4y} + \frac{28{\cdot}8}{6y} - \frac{1}{2}, \ldots (1)$$

$$\text{and } \frac{28{\cdot}8}{16x} + \frac{14{\cdot}4}{9x} + \frac{24}{12x} = \frac{24}{5y} + \frac{14{\cdot}4}{6y} + \frac{28{\cdot}8}{4y} - 1\tfrac{48}{60}, \ldots (2)$$

from (1), $(3{\cdot}2 + 0{\cdot}9 + 2)\dfrac{1}{x} = (4{\cdot}8 + 3{\cdot}6 + 4{\cdot}8)\dfrac{1}{y} - \dfrac{1}{2}$,

$$\text{or } \frac{6\cdot 1}{x} - \frac{13\cdot 2}{y} = -\frac{1}{2},$$

$$\text{or } \frac{61}{x} - \frac{132}{y} = -5. \quad \ldots\ldots \ (3)$$

Similarly, from (2), $\frac{3}{x} - \frac{8}{y} = -1, \quad \ldots\ldots \ (4)$

$$\left.\begin{aligned} &\text{from (3)}, \quad \frac{122}{x} - \frac{264}{y} = -10, \\ &\ldots\ldots \ (4), \quad \frac{99}{x} - \frac{264}{y} = -33, \end{aligned}\right\}$$

$$\therefore \ \frac{23}{x} = 23, \qquad \therefore \ x = 1.$$

$$\text{And } \ \frac{8}{y} = \frac{3}{x} + 1 = 4, \quad \therefore \ y = 2.$$

$\therefore$ P's rate is 12 miles, and Q's 10 miles, per hour.

II. 1st. Suppose P to go in direction AB, then P will arrive at C in $\frac{24}{12x} + \frac{14\cdot 4}{9x}$, or $3\cdot 6$, hours; and Q in $\frac{28\cdot 8}{4y}$, or $3\cdot 6$, hours, that is, they will meet at C.

2nd. Suppose Q to go in direction AB, then P arrives at C in $\frac{28\cdot 8}{9}$, or $3\cdot 2$, hours. Q reaches B in $\frac{24}{10}$, or $2\cdot 4$, hours, and would reach C in $\frac{14\cdot 4}{8}$, or $1\cdot 8$, hours more, and $\therefore$ will meet P before that time, suppose at z miles from C; then

$$3\cdot 2 + \frac{z}{16} = 2\cdot 4 + \frac{14\cdot 4 - z}{8},$$

$$\frac{3z}{16} = 2\cdot 4 + 1\cdot 8 - 3\cdot 2 = 1,$$

$$\therefore \ z = \frac{16}{3} = 5\tfrac{1}{3}.$$

7. Suppose that in the course of any one year the number of births in Ireland is on an average 32 for 1000 living in the island at the commencement of that year, the number of deaths and emigrations to the colonies, 21 in 1000, and of migrations to England 1 in 100. In England suppose the number of births in the course of any one year to be 3 for every 100 inhabitants living at the beginning of the year, the number of deaths and emigrations to the colonies 289 in 10,000, and of migrations to

Ireland 1 in 10,000. If the number of inhabitants in England was twice the number in Ireland at the beginning of 1850, in what year will the population of the former be three times that of the latter, according to the law above stated?

Let p = amount of increase per annum per head in Ireland,

$$=1+\frac{32}{1000}-\frac{21}{1000}-\frac{1}{100}=1+\frac{1}{1000}.$$

P = similar amount for England,

$$=1+\frac{3}{100}-\frac{289}{10,000}-\frac{1}{10,000}=1+\frac{1}{1000}=p,$$

B = migrations from England to Ireland,

$$=\frac{1}{10,000}, \text{ per head per annum.}$$

b = ... Ireland to England $=\frac{1}{100}$ ....

N = N°. of inhabitants in England at the beginning of 1850,

n = ... Ireland ....

N_x = ... England ....1850+x.

n_x = ... Ireland ....

then

$$n_1=pn+BN,$$
$$n_2=p(pn+BN)+B(PN+bn),$$
$$=p^2n+pBN+PBN+Bbn,$$
$$=p^2n+2pBN+Bbn,$$
$$n_3=p^3n+2p^2BN+pBbn$$
$$+B(p^2N+2pbn+BbN),$$
$$=p^3n+3p^2BN+3pBbn+bB^2N,$$

$$N_1=PN+bn,$$
$$N_2=P(PN+bn)+b(pn+BN),$$
$$=P^2N+Pbn+pbn+BbN,$$
$$=p^2N+2pbn+BbN,$$
$$N_3=p^3N+2p^2bn+pBbN$$
$$+b(p^2n+2pBN+Bbn),$$
$$=p^3N+3p^2bn+3pBbN+Bb^2n,$$

and so on. Now to detect the *law* of these successive results, we have

$$n_2=\frac{n}{2}\{(p+\sqrt{Bb})^2+(p-\sqrt{Bb})^2\}+\frac{N}{2}\sqrt{\frac{B}{b}}\{(p+\sqrt{Bb})^2-(p-\sqrt{Bb})^2\},$$

$$=\left\{\frac{n}{2}+\frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p+\sqrt{Bb}\}^2+\left\{\frac{n}{2}-\frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p-\sqrt{Bb}\}^2.$$

Similarly

$$n_3=\left\{\frac{n}{2}+\frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p+\sqrt{Bb}\}^3+\left\{\frac{n}{2}-\frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p-\sqrt{Bb}\}^3;$$

so that n_x and N_x may be put down at once, observing that n_1, n_2, n_3, &c. and so likewise N_1, N_2, N_3, &c. differ from each other only in the indices of $p \pm \sqrt{Bb}$.

$$n_x = \left\{\frac{n}{2} + \frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p+\sqrt{Bb}\}^x + \left\{\frac{n}{2} - \frac{N}{2}\sqrt{\frac{B}{b}}\right\}\{p-\sqrt{Bb}\}^x,$$

$$\text{and } N_x = \left\{\frac{N}{2} + \frac{n}{2}\sqrt{\frac{b}{B}}\right\}\{p+\sqrt{Bb}\}^x + \left\{\frac{N}{2} - \frac{n}{2}\sqrt{\frac{b}{B}}\right\}\{p-\sqrt{Bb}\}^x.$$

Now $N=2n$, and $N_x = 3n_x$, to find x; and substituting the numerical values given above for B, b, P, p, we get from $N_x = 3n_x$,

$$\left(\frac{501}{500}\right)^x = \frac{26}{21}, \quad \therefore\ x = \frac{\log 26 - \log 21}{\log 501 - \log 500} = 107.$$

$$\text{Or thus, } \left(1 + \frac{1}{500}\right)^x = 1 + \frac{x}{500} + \frac{x(x-1)}{1\times 2}\cdot\frac{1}{(500)^2} + \text{\&c.} = 1 + \frac{5}{21},$$

from which, by approximation, $x = 106{\cdot}5$ nearly.

Hence the year required $= 1850 + x = 1957$.

IV.

1. $\dfrac{x^2+2x+2}{x+1} + \dfrac{x^2+8x+20}{x+4} = \dfrac{x^2+4x+6}{x+2} + \dfrac{x^2+6x+12}{x+3}$; find x.

$$x+1+\frac{1}{x+1}+x+4+\frac{4}{x+4} = x+2+\frac{2}{x+2}+x+3+\frac{3}{x+3},$$

$$\therefore\ \frac{1}{x+1}+\frac{4}{x+4} = \frac{2}{x+2}+\frac{3}{x+3},$$

$$\therefore\ \frac{4}{x+4}-\frac{3}{x+3} = \frac{2}{x+2}-\frac{1}{x+1},$$

$$\therefore\ \frac{x}{x^2+7x+12} = \frac{x}{x^2+3x+2}, \quad \therefore\ x=0.$$

$$x^2+7x+12 = x^2+3x+2,$$

$$4x = -10,$$

$$\therefore\ x = -2\tfrac{1}{2}.$$

2. $(5x^2+x+10)^2+(x^2+7x+1)^2 = (3x^2-x+5)^2+(4x^2+5x+8)^2$; find x.

$$(5x^2+x+10)^2-(4x^2+5x+8)^2 = (3x^2-x+5)^2-(x^2+7x+1)^2,$$

$$(9x^2+6x+18)(x^2-4x+2)=(4x^2+6x+6)(2x^2-8x+4),$$

$9x^2+6x+18=2(4x^2+6x+6),$	and $x^2-4x+2=0,$
$x^2-6x+6=0,$	$\therefore\ x=2\pm\sqrt{2}\ldots(1)$
$x^2-6x+9=3,$	

$$\therefore\ x=3\pm\sqrt{3}\ldots\ldots\ldots\ldots(2).$$

3. $(x^2+4x-2)^2+3=4x(3x^2+4)$; find x.

$$x^4+8x^3+16x^2-4x^2-16x+7=12x^3+16x,$$
$$x^4-4x^3+12x^2-32x+7=0,$$
$$(x^2-2x+1)^2+6(x^2-2x+1)=16x,$$
$$(x-1)^4+8(x-1)^2=2(x-1)^2+16(x-1)+16,$$
$$(x-1)^4+8(x-1)^2+16=2\{(x-1)^2+8(x-1)+16\},$$
$$(x-1)^2+4=\sqrt{2}.(x-1+4),$$
$$(x-1)^2-\sqrt{2}(x-1)+\frac{1}{2}=4\sqrt{2}-\frac{7}{2},$$
$$x-1-\frac{1}{\sqrt{2}}=\pm\frac{1}{\sqrt{2}}\sqrt{8\sqrt{2}-7},$$
$$\therefore\ x=\frac{1}{2}\{2+\sqrt{2}\pm\sqrt{16-7\sqrt{2}}\}.$$

4. $$\left.\begin{aligned} &3x+3y-z=3, \ \ldots\ldots\ldots\ldots\ldots(1)\\ &x^2+y^2-z^2=\frac{14-9z}{2}, \ \ldots\ldots\ldots(2)\\ &x^3+y^3+z^3=3xyz+\frac{17z+44}{4}; \ \ldots(3)\end{aligned}\right\}\ \text{find } x,\ y, \text{ and } z.$$

From (1), $3(x+y+z)=4z+3,\ \ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$

..... (2), $x^2+y^2+z^2=2z^2+7-\frac{9z}{2},\ \ldots\ldots\ldots\ldots\ldots(\beta)$

..... (3), $2(x^3+y^3+z^3-3xyz)=\frac{17z+44}{2},\ \ldots\ldots\ldots(\gamma)$

Mult. (α) by (β), and subtract (γ), then

$$x^3+y^3+z^3+3(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2)+6xyz=\&c.$$

$$\text{or } (x+y+z)^3=8z^3-12z^2+21+\frac{29z}{2}-\frac{17z}{2}-22,$$
$$=8z^3-12z^2+6z-1=(2z-1)^3,$$
$$\therefore\ x+y+z=2z-1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\delta).$$

from (a), $x+y+z=\frac{4}{3}z+1$, $\quad\therefore\ \frac{2}{3}z=2$, and $z=3$.

From (1), $x+y=2$, and from (2), $x^2+y^2=\frac{5}{2}$,

$$\therefore\ x-y=\pm 1,$$

$$\therefore\ x=1\tfrac{1}{2}, \text{ or } \frac{1}{2}, \text{ and } y=\frac{1}{2}, \text{ or } 1\tfrac{1}{2}.$$

5. A derives his income from a fixed rental, B from his profession, C from both. In the first year A pays as much income-tax as B and C together, but in the second year B's and C's professional incomes being doubled, B pays as much as C, which is $\frac{4}{5}$ths of what A pays; also the total amount of their incomes in the two years is £5500. Assuming that the income-tax is higher for a fixed rental than for professional income in the ratio 3 : 2, find the incomes of A, B, C in the first year.

Let x£ $=A$'s income the first year,

$y=B$'s ....

$z+u=C$'s ...., u being the professional part,

then x, $2y$, $z+2u$, are their respective incomes the 2nd year.

Let $3r$ and $2r$ be income-tax per £1 for fixed rental and professional income respectively; then, by the question,

$$\left.\begin{aligned} 3x &= 2y+3z+2u, \ \ldots\ldots\ldots\ldots (1)\\ 4y &= 3z+4u=\frac{4}{5}.3x, \ \ldots\ldots\ldots (2)\\ \text{and }\ 2x&+3y+2z+3u=5500, \ldots\ldots (3)\end{aligned}\right\}$$

From (2), $3x=5y$, from (1), $\left.\begin{aligned}3y&=3z+2u,\\ \ldots\ (2),\ 4y&=3z+4u,\end{aligned}\right\}$

$$\therefore\ y=2u, \text{ and } z=\frac{4}{3}u, \text{ and } x=\frac{10u}{3},$$

$$\therefore \text{ from (3), } \frac{28u}{3}+9u=5500,$$

$$\therefore\ u=300,\ x=1000,\ y=600,\ z=400.$$

$\therefore$ required incomes are £1000, £600, and £700.

6. A and B start at the same time in a boat-race; A has 100 yards' start and has to row to a post D, B to a post C. At first A's rate : B's :: 40 : 39, but when the distance between A and B is $\frac{1}{6}$th of the remaining distance A has to row, A's speed is diminished in the ratio 79 : 80, so that two minutes after-

wards the distance between them is three yards more than half the remaining distance B has to row. At this point of B's course, his rate which has hitherto been uniform is increased eight yards a minute, while A's is still further diminished six yards a minute; and in one minute more B arrives at the post C, A being then three yards from the post D. Find the distance between C and D.

Let $40x = A$'s rate at first in yards *per minute*,

$39x = B$'s ....

$y =$ N°. of minutes occupied by B in the whole of his course,

$\therefore\ y-3 =$ .... A before he changes his speed,

$39xy+8 =$ *length* of B's course in yards,

$$40x(y-3)+2\times\frac{79x}{2}+\left(\frac{79x}{2}-6\right)+3 = \text{length of } A\text{'s course.}$$

Now dist. between A and B, when A first changes his speed, $= x(y-3)+100$; and dist. A has then to row $= 79x+\frac{79x}{2}-3$,

$$\therefore\ x(y-3)+100 = \frac{1}{6}\left(79x+\frac{79x}{2}-3\right) = \frac{1}{2}\left(\frac{79x}{2}-1\right)\text{.........(1).}$$

Two min. afterwards dist. between them is $x(y-2)+100$, and the dist. B has then to row $= 39x+8$,

$$\therefore\ x(y-2)+100 = \frac{1}{2}(39x+8)+3,\text{......................(2).}$$

From (1), $x(y-2)+100 = \frac{1}{2}\left(\frac{83x}{2}-1\right)$,

$$\therefore\ \frac{5x}{2} = 15,\quad \therefore\ x = 6.\quad \text{Also from (2), } y = 6.$$

$\therefore$ length of A's course $= 240\times3+3\times79\times3-3 = 1428$ yards,

.... B's .... $= 39\times36+8 = 1412$ yards,

$\therefore$ dist. between C and $D = 1428-1412+100 = 116$ yards.

7. Three men, A, B, C, walk in the same direction in the circumferences of three concentric circles, starting simultaneously from points where they are at their least distances from each other. A walks his circuit in an even number of hours, (greater than four), B and C their circuits in one hour and two hours less respectively. Whenever A and B are at there greatest distance from each other, they alter their rates in such a manner, that the times they would take to walk their circuits at the rates they are *then* going are interchanged; and whenever A and C are *again* at

their least distance their times are interchanged in a similar manner. When A and B are at their greatest distance the first time, A has walked a distance equal to twenty-two times C's circuit; and when they are at their greatest distance the third time, B has walked a distance equal to forty-two times A's circuit, and C has then walked ten miles less than forty times B's circuit, and is at his least distance from B. Required the rates of A, B, C, at first.

Let $x+1$, x, $x-1$, be the N°. of hours in which A, B, C, will go their rounds at first;

y, z, u, the three circumferences of A, B, C, in miles;

then, since B goes his round in less time than A, B will have gained half a circumference* on A when A and B are at their *greatest* dist., and this will be in $\frac{1}{2}x(x+1)$ hours. Again, A and C will be at their *least* dist. when C has gained a ○ on A, which will be in $\frac{1}{2}(x-1)(x+1)$ hours; $\therefore$ A and C are at their *least* dist. *before* A and B are at their *greatest.* Also, since A and C are at their *least* dist. every $\frac{1}{2}(x-1)(x+1)$ hours (so long as they retain the times $x+1$, $x-1$, respectively), B will in this time gain on A, (so long as A retains his 1st speed), $\frac{x-1}{2x}$ of a ○, for he gains the whole ○ in $x(x+1)$ hours; and A will in the time $\frac{1}{2}(x-1)(x+1)$ gain on B, (after A has assumed the time $x-1$ for his circuit), $\frac{x+1}{2x}$ of a ○, for he gains the whole ○ on B in $x(x-1)$ hours.

$\therefore$ when A and C are *again* at *least* dist. 1st time

....	B is	$\frac{x-1}{2x}$ of a ○ a-head of	A,
......................2nd time	A is $\frac{x+1}{2x}-\frac{x-1}{2x}$ or	$\frac{1}{x}$	B,
......................3rd......	B is $\frac{x-1}{2x}-\frac{1}{x}$ or	$\frac{x-3}{2x}$	A,
......................4th......	A is $\frac{x+1}{2x}-\frac{x-3}{2x}$ or	$\frac{2}{x}$	B,
......................5th......	B is $\frac{x-1}{2x}-\frac{2}{x}$ or	$\frac{x-5}{2x}$	A,
......................6th......	A is $\frac{x+1}{2x}-\frac{x-5}{2x}$ or	$\frac{3}{x}$	B,

and so on.

Hence, the dist. B is a-head of A, when A and C are at

* As this word will occur often let it be represented by ○.

their *least* dist. for the 1st, 3rd, 5th, &c. time, is always less than half a ○, and keeps decreasing; but the dist. A is a-head of B, when A and C are at their least dist. for the 2nd, 4th, 6th, &c. time, keeps increasing. Also when A and C are at their least dist. for the x^{th} time, ($\because$ x is odd), the dist. B is a-head of $A=0$, and at the $(x+1)^{\text{th}}$ time A is $\frac{x+1}{2x}$ of a ○, (which $> \frac{1}{2}$○), a-head of B; $\therefore$ in the interval between the x^{th} and $(x+1)^{\text{th}}$ times of A and C being at their *least* dist. A and B will be at their *greatest* dist. for the *first* time.

Now, since after the x^{th} time of A and C being at least dist. A goes his round in $(x-1)$ hours, and B in x hours, $\therefore$ A gains half of a ○ on B in $\frac{1}{2}x(x-1)$ hours, that is, A and B will be at their *greatest* dist. for the *first* time $\frac{1}{2}x(x-1)$ hours after A and C are at their *least* dist. for the x^{th} time. *Previous* to A and B being at their greatest dist., A would gain on C a whole ○ in $\frac{1}{2}(x-1)(x+1)$ hours, $\therefore$ when A and B are *at* their greatest dist. he will have gained $\frac{x}{x+1}$ of a ○, and he still *gains* on C *after* A and B are at their greatest dist., because he now goes round in x, and C in $(x+1)$, hours. Now, when A and B are *at* greatest dist. the part of a ○ which A has to gain on C is $1-\frac{x}{x+1}$, or $\frac{1}{x+1}$; and he would now gain a ○ in $x(x+1)$ hours, $\therefore$ he gains $\frac{1}{x+1}$ of a ○ in x hours, that is, A and C are at their least dist. the $(x+1)^{\text{th}}$ time x hours after A and B are at their greatest dist. the *first* time.

The times now taken by A, B, C, are $x+1$, $x-1$, x; and before A and C are at least dist. the $(x+1)^{\text{th}}$ time, the times of circuit were x, $x-1$, $x+1$, $\therefore$ B would gain a ○ on A in $x(x-1)$ hours, and $\therefore$ in x hours he will have gained $\frac{1}{x-1}$ of a ○. But A was half a ○ a-head of B x hours before, $\therefore$ A will now be $\frac{1}{2}-\frac{1}{x-1}$, or $\frac{x-3}{2(x-1)}$ of a ○ a-head of B; that is, when A and C are at least dist. the $(x+1)^{\text{th}}$ time, A is $\frac{x-3}{2(x-1)}$ of a ○ a-head of B; $\therefore$ B will have to gain $\frac{1}{2}+\frac{x-3}{2(x-1)}$, or $\frac{x-2}{x-1}$ of a ○, before A and B are at greatest dist. again; and since B gains a ○ in $\frac{1}{2}(x+1)(x-1)$ hours, $\therefore$ he will gain $\frac{x-2}{x-1}$ in $\frac{1}{2}(x-2)(x+1)$ hours, which $< x(x+1)$, the time A and C take to be at their *least* dist. again; $\therefore$ $\frac{1}{2}(x-2)(x+1)$ hours after A and C are at their least dist.

the $(x+1)^{\text{th}}$ time A and B are at their *greatest* dist. the *second* time.

Previous to this the times of A, B, C, were $x+1$, $x-1$, x, $\therefore$ C would gain a $\bigcirc$ on A in $x(x+1)$ hours, and $\therefore$ in $\frac{1}{2}(x-2)(x+1)$ hours he has gained $\frac{x-2}{2x}$ of a $\bigcirc$; and *after* A and B are at greatest dist. the *second* time, the times of A, B, C, are $x-1, x+1, x$; $\therefore$ A has to gain on C $\frac{x-2}{2x}$ of a $\bigcirc$, before A and C are again at the least distance; and he gains a whole $\bigcirc$ in $x(x-1)$ hours, $\therefore$ he gains $\frac{x-2}{2x}$ of a $\bigcirc$ in $\frac{1}{2}(x-1)(x-2)$ hours, which $<\frac{1}{2}(x-1)(x+1)$, the time A and B take to be at greatest dist. the 3rd time. $\therefore$ $\frac{1}{2}(x-1)(x-2)$ hours after A and B are at their greatest dist. the *second* time, A and C will be at least dist. the $(x+2)^{\text{th}}$ time. Now, previous to this the times of A, B, C, were $x-1$, $x+1$, x, $\therefore$ A would gain a $\bigcirc$ on B in $\frac{1}{2}(x-1)(x+1)$ hours, and $\therefore$ in $\frac{1}{2}(x-1)(x-2)$ hours he will have gained $\frac{x-2}{x+1}$ of a $\bigcirc$, $\therefore$ since, when A and B were at greatest dist. the 2nd time, B was half of a $\bigcirc$ a-head of A, A is now a-head of B $\frac{x-2}{x+1}-\frac{1}{2}$, or $\frac{x-5}{2(x+1)}$ of a $\bigcirc$. But after A and C are at least dist. the $(x+2)^{\text{th}}$ time, the times of A, B, C, are x, $x+1$, $x-1$, $\therefore$ A would gain a $\bigcirc$ on B in $x(x+1)$ hours, and he has to gain $\frac{1}{2}-\frac{x-5}{2(x+1)}$, or $\frac{3}{x+1}$ of a $\bigcirc$, before A and B are at greatest dist. the *third* time. This will take him $3x$ hours, which $<x(x-1)$, the number of hours A and C take to be at least dist. the $(x+3)^{\text{th}}$ time; and in $3x$ hours C will have gained $\frac{3}{x-1}$ of a $\bigcirc$ on A......(α).

Now, when A and C have been x times at least dist., A has walked $\frac{y}{x+1}\cdot\frac{(x-1)(x+1)}{2}+\frac{y}{x-1}\cdot\frac{(x-1)(x+1)}{2}+$ &c., to x terms,

or $\frac{y}{2}\{(x-1)+(x+1)+(x-1)+\text{\&c. to } x \text{ terms}\}$, or $\frac{y}{2}(x^2-1)$, $\because$ x is odd,

$\therefore$ dist. A has walked when A and B are greatest dist. the 1st time $=\frac{y}{2}(x^2-1)+\frac{y}{x-1}\cdot\frac{1}{2}x(x-1)$, or $\frac{y}{2}(x^2+x-1)$,

$$\therefore \text{ by the question, } \frac{y}{2}(x^2+x-1)=22u,\ldots\ldots(1).$$

Again, dist. B has walked when A and C are at least dist. the x^{th} time

$$=\frac{z}{x}\cdot x\frac{(x-1)(x+1)}{2}, \text{ or } \frac{z}{2}(x-1)(x+1);$$

also, dist. B has walked when A and B are at greatest dist. the 3rd time

$$=\frac{z}{2}(x-1)(x+1)+\frac{z}{x}\cdot\frac{1}{2}x(x-1)+\frac{z}{x-1}\cdot x+\frac{z}{x-1}\cdot\frac{(x+1)(x-2)}{2}$$

$$+\frac{z}{x+1}\cdot\frac{(x-1)(x-2)}{2}+\frac{z}{x+1}\cdot 3x,\text{ or }\frac{z}{2}(x^2+3x+2),$$

$$\therefore\text{ by the question, }\frac{z}{2}(x^2+3x+2)=42y\ldots\ldots(2)$$

Again, when A and B are at greatest distance the 3rd time, C has walked

$$\frac{u}{2}(x^2+1)+\frac{u}{x+1}\cdot\frac{x(x-1)}{2}+\frac{u}{x+1}\cdot\frac{(x+1)(x-2)}{2}+\frac{u}{x}\cdot\frac{(x-1)(x-2)}{2}+\frac{u}{x-1}\cdot 3x,$$

$$\text{or }\frac{u}{2}(x^2+3x-3)+\frac{3ux}{x-1},$$

$$\therefore\text{ by the question, }\frac{u}{2}(x^2+3x-3)+\frac{3ux}{x-1}=40z-10\ldots\ldots(3).$$

Again, $\because$ C is at least distance from B, when A and B are at greatest distance the 3rd time, $\therefore$ C is at his greatest distance from A, and $\therefore$ from (α), $\dfrac{3}{x-1}=\dfrac{1}{2}$, and $\therefore$ $x=7$.

$\therefore$ (1) becomes $55y=44u$, or $5y=4u$,

(2) .... $36z=42y$, or $6z=7y$,

(3) .... $\dfrac{67u}{2}+\dfrac{7u}{2}=40z-10$, or $37u=40z-10$,

from (1) and (2), $30z=28u$, $\therefore$ $40z=\dfrac{112u}{3}$,

$$\therefore\ 37u=\frac{112u}{3}-10,\ \therefore\ u=30,\ y=24,\ z=28,$$

A's rate at first $=\dfrac{y}{x+1}=3$ miles per hour,

B's .... $=\dfrac{z}{x}=4$ ....

C's .... $=\dfrac{u}{x-1}=5$............

[This singularly complex Problem and its solution are both due to the ingenuity of Mr. Bower, late Fellow of St John's College. It is to be hoped that some shorter and simpler solution may yet be hit upon; but it is no small trial of patience and skill to get any solution at all. T. L.]

V.

1. $(a+b)^2x+\left(\frac{ab}{a-b}\right)^2\cdot\left(4-\frac{3}{x}\right)=2ab$; find x.

$$\frac{(a+b)^2}{ab}\cdot x+\frac{ab}{(a-b)^2}\cdot\left(4-\frac{3}{x}\right)=2,$$

$$\left(\frac{a^2+b^2}{ab}+2\right)x+\frac{1}{\frac{a^2+b^2}{ab}-2}\cdot\left(4-\frac{3}{x}\right)=2,$$

$$2(c+1)x+\frac{1}{2(c-1)}\cdot\frac{4x-3}{x}=2,\ \text{if}\ \frac{a^2+b^2}{ab}=2c,$$

$$4(c^2-1)x^2+4x-3=4cx-4x,$$

$$4c^2x^2-4cx+1=4x^2-8x+4,$$

$$2cx-1=\pm(2x-2),$$

$$\therefore\ x=\frac{3}{2(1+c)},\ \text{or}\ \frac{1}{2(1-c)},$$

$$=\frac{3ab}{(a+b)^2},\ \text{or}\ \frac{-ab}{(a-b)^2}.$$

2. $(x+2\sqrt{x})^{\frac{1}{2}}-(x-2\sqrt{x})^{\frac{1}{2}}=2(x^2-4x)^{\frac{1}{8}}$; find x.

Squaring, $2x-2(x^2-4x)^{\frac{1}{2}}=4(x^2-4x)^{\frac{1}{4}}$,

$$(x^2-4x)^{\frac{1}{2}}+2(x^2-4x)^{\frac{1}{4}}+1=x+1,$$

$$(x^2-4x)^{\frac{1}{4}}+1=\pm(x+1)^{\frac{1}{2}},$$

$$(x^2-4x)^{\frac{1}{2}}=x+2\mp2(x+1)^{\frac{1}{2}},$$

$$x^2-4x=x^2+8x+8\mp4(x+2)\sqrt{x+1},$$

$$4(x+1)\mp2(x+2)\sqrt{x+1}+\tfrac{1}{4}(x+2)^2=\tfrac{1}{4}(x-2)^2,$$

$$2\sqrt{x+1}=\pm\tfrac{1}{2}(x+2)\pm\tfrac{1}{2}(x-2)=\pm x,\ \text{or}\ \pm2,$$

$$\therefore\ 4x+4=x^2,\ \text{or}\ 4,$$

$$\therefore\ x=0,\ \text{or}\ 2(1\pm\sqrt{2}).$$

3. $(12x-1)(6x-1)(4x-1)(3x-1)=\frac{1}{96}$; find x.

$$(36x^2-15x+1)(24x^2-10x+1)=\frac{1}{96},$$

$$\{3(12x^2-5x)+1\}\{2(12x^2-5x)+1\}=\frac{1}{96},$$

$$\text{or } (3y+1)(2y+1)=\frac{1}{96}, \text{ if } 12x^2-5x=y,$$

$$6y^2+5y+1=\frac{1}{96},$$

$$36y^2+5\times 6y-\left(\frac{5}{2}\right)^2=\frac{1}{16}+\frac{1}{4}=\frac{5}{16},$$

$$\therefore\ 6y=\frac{\pm\sqrt{5}}{4}-\frac{5}{2}, \text{ or } 3y=\frac{\pm\sqrt{5}-10}{8},$$

$$\therefore\ 36x^2-\frac{5}{2}.6x+\left(\frac{5}{4}\right)^2=\frac{5\pm 2\sqrt{5}}{16},$$

$$\therefore\ 6x=\frac{5\pm\sqrt{5\pm 2\sqrt{5}}}{4},$$

$$\therefore\ x=\frac{1}{24}(5\pm\sqrt{5\pm 2\sqrt{5}}).$$

4. $$\left.\begin{aligned}\frac{x+y}{4}&=\frac{xy}{x+y}+\frac{1}{2}, \ \ldots\ldots(1)\\ x^2+y^2&=\frac{8x^2y^2}{(x+y)^2}+22, \ \ldots(2)\end{aligned}\right\}\text{ find } x \text{ and } y.$$

Let $x+y=v$, and $x-y=z$,

then $x=\frac{1}{2}(v+z)$, and $y=\frac{1}{2}(v-z)$,

$\therefore$ from (1), $$\frac{v}{4}=\frac{v^2-z^2}{4v}+\frac{1}{2},$$

$$v^2=v^2-z^2+2v, \quad \therefore\ z^2=2v\ldots\ldots(3).$$

From (2), $$\tfrac{1}{4}(v+z)^2+\tfrac{1}{4}(v-z)^2=\frac{(v^2-z^2)^2}{2v^2}+22,$$

$$v^4+v^2z^2=v^4-2v^2z^2+z^4+44v^2,$$

$$3v^2z^2=z^4+44v^2,$$

$\therefore$ by (3), $$6v^3=4v^2+44v^2=48v^2,$$

$$\therefore\ v=8, \quad \therefore\ z^2=16, \text{ and } z=\pm 4.$$

$$\therefore\ x+y=8, \quad x-y=\pm 4,$$

$$\therefore\ x=6, \text{ or } 2, \text{ and } y=2, \text{ or } 6.$$

5. *A* and *B* set out at the same time from the same place, and walk in the same direction. Whenever the distance between them is an even number of miles, *A* increases his speed $\frac{1}{4}$ mile

per hour; and when it is an odd number of miles, B increases his speed $\frac{1}{2}$ mile per hour. When A is 4 miles in advance, B has walked $30\frac{2}{3}$ miles, and A has walked $1\frac{1}{3}$ miles more than he would have done in the same time, had he walked uniformly at his first rate. Required the rates of A and B at starting.

Let x and y be the miles per hour of A and B at starting,

t_1 = hours between starting and time when A is 1 mile a-head,

t_2 = ... time when A is 1 mile a-head and when 2,

t_3 = ...2 miles 3,

t_4 = ...3 .... 4;

then A's rate during t_1, t_2, is x miles per hour,

$$\dots\dots\dots\dots\; t_3,\ t_4,\ \dots\ x+\frac{1}{4}\dots\dots\dots,$$

B's rate during t_1 is y miles per hour,

$$\dots\dots\dots\dots t_2,\ t_3\dots y+\frac{1}{2}\dots\dots\dots$$

$$\dots\dots\dots\dots t_4\dots y+1\ \dots\dots\dots$$

$$\therefore \text{ by question, }\quad (t_3+t_4)\frac{1}{4}=1\tfrac{1}{3}\dots\dots\dots\dots(1)$$

$$\text{also}\quad (t_1+t_2+t_3+t_4)y+(t_2+t_3)\frac{1}{2}+t_4=30\tfrac{2}{3},\ \dots\dots(2)$$

$$\text{but}\quad (x-y)t_1=1,\quad \left(x-y-\frac{1}{2}\right)t_2=1,\quad \left\{x+\frac{1}{4}-\left(y+\frac{1}{2}\right)\right\}t_3=1,\ \text{and}$$

$$\left\{x+\frac{1}{4}-(y+1)\right\}t_4=1;$$

$$\therefore \text{ from (1), }\ \frac{1}{x-y-\frac{1}{4}}+\frac{1}{x-y-\frac{3}{4}}=\frac{16}{3};\ \text{write } z \text{ for } x-y,$$

$$2z-1=\frac{16}{3}\left(z^2-z+\frac{3}{16}\right),$$

$$z^2-\frac{11z}{8}=-\frac{3}{8},\quad \text{from which } z=1, \text{ or } \frac{3}{8}.$$

$$\text{Taking } z=1,\quad t_1=1,\quad t_2=2,\quad t_3=\frac{4}{3},\quad t_4=4,$$

$$\therefore \text{ from (2), }\ \frac{25}{3}y=25,\ \text{and } y=3,\quad \therefore x=4.$$

6. Three vessels, A, B, C, contain liquids in the proportion $3:2:1$, and their prices per gallon are as $1:2:3$. A certain quantity is poured from A into B, and the same quantity from

the mixture in B into C; the same quantity is now poured from the mixture in C into B, and from the new mixture in B into A. The value of the liquid in A is thereby increased in the ratio of $35 : 27$; but had the quantity poured out each time been one gallon more than it was, its value would have been increased in the ratio of $3 : 2$. Find the quantity of liquid in each vessel.

Let $3x$, $2x$, x, be the number of gallons in A, B, C, at first,

y, $2y$, $3y$, the prices per gallon of the liquids,

z, the number of gallons taken out of each;

then 1st, A becomes $3x-z$, in quantity, and $(3x-z)y$ in value,

B .... $2x+z$, ..., ... $(4x+z)y$ ....

then C .... $x+z$, ..., ... $3xy+\frac{z}{2x+z}(4x+z)y$, or

$$\frac{(6x+z)(x+z)}{2x+z}.y \ldots\ldots$$

and, at the same time, B is again reduced to $2x$ in quantity, and to

$$\frac{2x}{2x+z}.(4x+z)y, \text{ or } \frac{y}{2x+z}\{8x^2+2xz\} \text{ in value.}$$

Next, B becomes again $2x+z$ in quantity, and

$$\frac{y}{2x+z}\{8x^2+2xz\}+\frac{z}{x+z}.\frac{(6x+z)(x+z)}{2x+z}.y, \text{ or}$$

$$\frac{y}{2x+z}\{8x^2+8xz+z^2\} \text{ in value.}$$

A then becomes $3x$ in quantity, and *in value*

$$(3x-z)y+\frac{z}{2x+z}.\frac{y}{2x+z}.\{8x^2+8xz+z^2\},$$

$$\text{or } y\left\{3x+4xz.\frac{x+z}{(2x+z)^2}\right\}, \text{ by reduction;}$$

$$\therefore \text{ by the question, } 1+\frac{4}{3}z.\frac{x+z}{(2x+z)^2}=\frac{35}{27}\ldots\ldots\ldots(1)$$

$$\text{or } 7z^2+xz-8x^2=0,$$

$$\text{or } (7z+8x)(z-x)=0, \quad \therefore x=z.$$

Again, putting $z+1$, or u, for z, and $\frac{3}{2}$ for $\frac{35}{27}$ in (1), we have, in the same manner,

$$(5u+6)(u-2)=0, \quad \therefore u=2, \text{ or } z+1=2, \quad \therefore x=1,$$

and the number of gallons required are 3, 2, 1.

7. Reckoning meat by the stone, wheat by the quarter, and hay by the load, the price of hay at first was equal to that of wheat together with 4 times that of meat. The rise or fall of meat is $\frac{1}{6}$th of the rise or fall of wheat, together with $\frac{1}{16}$th that of hay, except during the first month, when from scarcity of fodder, hay rose 8*s*., and the effect on meat from this cause alone was a fall of 6*d*. The *variation* in hay was 8*s*. each month, and wheat rose 2*s*. a month from the first, until, after a certain number of months, there was the same relation between the prices as at first; wheat being now 50*s*. and remaining stationary. After as many months more, hay was 12 times the price of meat. Required the prices of wheat and hay at first, that of meat being 6*s*.

Let the prices at first of meat, wheat, and hay, respectively, be 6, x, and $x+24$, in shillings;

let n = number of months in which wheat rises to 50*s*., then

$$x+2n=50, \text{ or } n=25-\frac{x}{2} \ldots\ldots (1)$$

let $8(y+1)$ = the rise or fall upon hay in n months,

$8z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 2n$ months,

where y and z are integral; then by the question,

$$x+24+8(y+1)=50+4\left(6+\frac{2n}{6}+\frac{8y}{16}-\frac{1}{2}\right), \ \ldots\ldots\ldots (2)$$

$$\text{also} \quad x+24+8(y+z+1)=12\left(6+\frac{2n}{6}+\frac{8y}{16}-\frac{1}{2}+\frac{8z}{16}\right)\ldots\ldots (3).$$

$$\left.\begin{aligned} &\text{From (1) and (2),} \quad 5x+18y=220\ldots\ldots(4) \\ &\text{also from (1) and (3),} \quad 3x+2y+2z=134\ldots(5) \end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{from (4),} \quad 15x+54y=660, \\ &\ldots\ldots (5), \quad 15x+10y+10z=670, \end{aligned}\right\}$$

$$\therefore\; 10z-44y=10,$$

$$\text{or } 5z-22y=5, \ldots\ldots (6)$$

$$\therefore \left.\begin{aligned} z&=1+22t, \\ y&=5t, \end{aligned}\right\} \text{ is the \textit{general} solution of (6),}$$

and from (4), $\therefore\; x=44-18t$; and $n=9t+3$.

Let $t=0$, then $x=44$, the required price of wheat, and $x+24 = 68$, the required price of hay.

To shew that this is the *only* solution, we have from (1) and (2), $n=\frac{9y}{5}+3=9t+3$, $\therefore$ t cannot have a *negative* value, for then n would be negative, which is not allowable, by the question. The only other values of t which will permit x to be positive are $t=1$, and $t=2$, both of which make n less than z, whereas we know that n is never less than z.

VI.

1. $\frac{1}{5}\cdot\frac{(x+1)(x-3)}{(x+2)(x-4)}+\frac{1}{9}\cdot\frac{(x+3)(x-5)}{(x+4)(x-6)}-\frac{2}{13}\cdot\frac{(x+5)(x-7)}{(x+6)(x-8)}=\frac{92}{585}.$

Let $(x-1)^2=y$; then the equation becomes

$$\frac{1}{5}\cdot\frac{y-4}{y-9}+\frac{1}{9}\cdot\frac{y-16}{y-25}-\frac{2}{13}\cdot\frac{y-36}{y-49}=\frac{92}{585}.$$

Now $\frac{1}{5}+\frac{1}{9}-\frac{2}{13}=\frac{92}{585}.$

$\therefore$ subtracting corresponding terms,

$$\frac{1}{y-9}+\frac{1}{y-25}-\frac{2}{y-49}=0,$$

$$\text{or } \frac{1}{y-9}-\frac{1}{y-49}+\frac{1}{y-25}-\frac{1}{y-49}=0,$$

$$\text{or } \frac{40}{y-9}+\frac{24}{y-25}=0,$$

$$\text{or } 5y-125+3y-27=0;$$

$$\therefore 8y=152, \text{ or } y=19;$$

$$\therefore x=1\pm\sqrt{19}.$$

2. $\left(\frac{x+6}{x-6}\right)\left(\frac{x-4}{x+4}\right)^2+\left(\frac{x-6}{x+6}\right)\left(\frac{x+9}{x-9}\right)^2=2\cdot\frac{x^2+36}{x^2-36}.$

The right-hand member $=\frac{x+6}{x-6}+\frac{x-6}{x+6}$;

$$\therefore \left(\frac{x-6}{x+6}\right)\cdot\left\{\left(\frac{x+9}{x-9}\right)^2-1\right\}=\left(\frac{x+6}{x-6}\right)\left\{1-\left(\frac{x-4}{x+4}\right)^2\right\};$$

$$\therefore \frac{x-6}{x+6}\cdot\frac{36x}{(x-9)^2}=\frac{x+6}{x-6}\cdot\frac{16x}{(x+4)^2};$$

$$\therefore x=0, \text{ or } \left(\frac{x-6}{x+6}\right)^2=\frac{4}{9}\cdot\left(\frac{x-9}{x+4}\right)^2;$$

$$\therefore \frac{x-6}{x+6} = \pm\frac{2}{3}\cdot\frac{x-9}{x+4};$$

$$\therefore 3(x^2-2x-24) = \pm 2(x^2-3x-54);$$

the upper sign makes x impossible;

the lower sign gives $5x^2-12x = 180$;

$$\therefore x = \frac{6}{5}(1 \pm \sqrt{26}).$$

3. $(x-1)^2+(a-1)^2 = 2(ax+1)+\sqrt{3(x+a)^2+4ax}$; find x.

$$\therefore x^2+a^2-2ax-2(x+a) = \sqrt{(3x+a)(x+3a)};$$

$$\therefore x+3a+2\sqrt{(3x+a)(x+3a)}+3x+a = 2(x-a)^2;$$

$$\therefore \sqrt{x+3a}+\sqrt{3x+a} = \pm\sqrt{2}\,.\,(x-a),$$

$$= \pm\frac{1}{\sqrt{2}}\{(3x+a)-(x+3a)\}\ldots\ldots\ldots\ldots(\alpha).$$

This resolves itself into the two equations,

$$\sqrt{3x+a}+\sqrt{x+3a} = 0,$$

(from which $x = a$, which satisfies the original equation, if negative sign be taken before the root), and

$$\sqrt{3x+a}-\sqrt{x+3a} = \pm\sqrt{2}\ldots\ldots\ldots\ldots\ldots\ldots(\beta):$$

adding (α) and (β), we have

$$\pm\sqrt{2}(x-a+1) = 2\sqrt{3x+a};$$

$$\therefore (x-a)^2+2(x-a)+1 = 6x+2a;$$

$$\therefore x = a+2 \pm \sqrt{8a+3}.$$

4. $\left.\begin{aligned} x^2+a^2+y^2+b^2 &= \sqrt{2}\{x(a+y)-b(a-y)\},\\ x^2-a^2-y^2+b^2 &= \sqrt{2}\{x(a-y)+b(a+y)\};\end{aligned}\right\}$ find x and y.

by addition, $x^2+b^2 = \sqrt{2}(ax+by)$,

by subtraction, $y^2+a^2 = \sqrt{2}(xy-ab)$;

multiplying these together,

$$(x^2+b^2)(y^2+a^2) = 2(ax+by)(xy-ab);$$

$$\therefore (xy-ab)^2+(ax+by)^2 = 2(ax+by)(xy-ab);$$

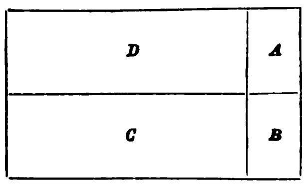
$$\therefore xy-ab = ax+by, \text{ or } y = a\frac{x+b}{x-b};$$

$$\therefore\ x^2+b^2=\sqrt{2}\left(ax+ab.\frac{x+b}{x-b}\right)=a\sqrt{2}.\frac{x^2+b^2}{x-b};$$

$$\therefore\ x-b=a\sqrt{2},\ \text{or}\ x=a\sqrt{2}+b,$$

$$\text{and}\ y=a\frac{x+b}{x-b}=b\sqrt{2}+a.$$

5. A rectangular field is divided by two lines parallel to two of its adjacent sides into four parts, of which the least has its longer side in the shorter side of the field. The ratio of the perimeters of the greatest and least parts is 4 : 1, and that of the other two is 3 : 2. Had however the sides of the least part been double their present lengths, the ratio of the areas of those parts which would *then* have been greatest and least would have been 4 : 1, and of the other two, one would have contained an acre more than the other. Find the number of acres in the field.



Let x = longer side of least part (A) in yards,

y = semi-perimeter of the field.

Then since the greatest and least parts are *opposite*, the sum of their perim^{rs}. = perim^r. of field.

$\therefore$ by question,

$$\frac{1}{2}\ \text{perim}^r.\ \text{of}\ A=\frac{y}{5},\ \text{of}\ C=\frac{4y}{5},\ \text{of}\ B=\frac{2y}{5},\ \text{of}\ D=\frac{3y}{5};$$

$\therefore$ original sides of A are x and $\frac{y}{5}-x$,

.... B ... $\frac{y}{5}-x$...... $\frac{y}{5}+x$,

.... C ... $\frac{y}{5}+x$...... $\frac{3y}{5}-x$,

.... D ... x ...... $\frac{3y}{5}-x$;

and in the supposed case,

sides of A are $2x$ and $2\left(\frac{y}{5}-x\right)$,

.... B ... $2\left(\frac{y}{5}-x\right)$ ... $\frac{y}{5}$,

.... C ... $\frac{y}{5}$ ... $\frac{2y}{5}$,

.... D ... $2x$ ... $\frac{2y}{5}$.

Now, by question, $x > \frac{y}{5} - x$; $\therefore\ 2x > \frac{y}{5}$;

$$\therefore \text{ new area of } D > \text{that of } C,$$
$$\text{and } \ldots\ldots\ldots\ C > \ldots\ldots\ B.$$

Hence B and D are *now* the least and greatest parts;

$$\therefore \text{ by question, } 2x \times \frac{2y}{5} : 2\left(\frac{y}{5} - x\right)\frac{y}{5} :: 4 : 1,$$

$$\text{or } 2x = 4\left(\frac{y}{5} - x\right);\ \therefore\ y = \frac{15x}{2};$$

$\therefore$ the sides of A in supposed case are $2x$ and x,

$$\ldots\ldots\ldots\ C\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \frac{3x}{2} \ldots 3x;$$

$\therefore$ by question,

$\frac{9x^2}{2} - 2x^2 = 4840$, the No. of square yards in an acre;

$$\therefore\ x = 44;$$

$$\therefore \text{ sides of the field are } \frac{7x}{2} \text{ and } 4x;$$

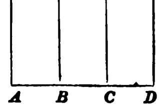
$$\therefore \text{ area} = 14x^2 = \frac{14(44)^2}{4840} \text{ acres} = 5\tfrac{3}{5} \text{ acres}.$$

6. Four points A, B, C, D, move uniformly with velocities in Geometrical Progression in four equidistant and equal parallel lines, whose extremities are situated in two parallel lines from one of which they start together, and when they reach the other extremities, they return, and so on continually. After a certain interval B, C, D, are in a straight line for the first time; after twice that interval A, B, C, are in a straight line for the first time; 36 seconds after this, A, C, D, are in a straight line for the second time, and the space that has been passed over by B is 14 inches more than that passed over by A, when A, B, C, D, are again in a straight line. Required the velocities of A, B, C, D.

Let x, rx, r^2x, r^3x be the velocities of A, B, C, D, in inches per second,

$$y = \text{length of each line},$$

then the consideration that B, C, D, are in a straight line *before* A, B, C, are in a straight line, shews that the velocities of A, B, C, D, are in *increasing* order; and it is clear that B, C, D, will be first in a line when B and C are moving forward, and D is on its first return.



Let t be the time (in seconds) from starting when this takes place, then the distances of B, C, D, from the starting points are trx, tr^2x, $2y-tr^3x$; and since they are in a straight line, we have

$$tr^2x-trx=(2y-tr^3x)-tr^2x;$$

$$\therefore\ txr(r^2+2r-1)=2y\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, to find when A, B, C are in a straight line, change t into $2t$, and x into $\frac{x}{r}$;

$$\therefore\ 2tx(r^2+2r-1)=2y\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, (1) and (2) shew that $r=2$, that is, the velocities of A, B, C, D, are x, $2x$, $4x$, $8x$.

Now the velocities of A, C, D, being as the Nos. 1, 4, 8, a little consideration will shew that they are a *second* time in a straight line, when C is on its first return from F, and D is a second time moving towards G; $\therefore$ if t_1 be the time from starting when A, C, D, are a *second* time in a straight line, we have the distances of A, C, D, from original starting points

$$t_1x,\quad 2y-t_14x,\quad t_18x-2y;$$

$\therefore$ since they are in a straight line,

$$(2y-t_14x)-t_1x=2\{(t_18x-2y)-(2y-t_14x)\};$$

$$\therefore\ t_1.29x=10y;\ \therefore\ t_1=\frac{10y}{29x}.$$

Now, by question, $t_1=2t+36$, and from (1), $t=\frac{y}{7x}$;

$$\therefore\ \frac{10y}{29x}=\frac{2y}{7x}+36\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, from the relative velocities of A, B, C, D, it is easily seen, that they will first be in a straight line again when A has moved over $\frac{2y}{3}$;

$$\therefore\ \text{by question, } \frac{2y}{3}+14=\frac{10y}{29x}\times 2x;$$

$$\therefore\ y\left(\frac{10}{29}-\frac{1}{3}\right)=7;\ \therefore\ y=3\times 7\times 29;$$

$$\therefore\ \text{from (3), } \frac{30\times 7}{x}=\frac{6\times 29}{x}+36;$$

$$\therefore\ \frac{36}{x}=36;\ \therefore\ x=1;$$

$\therefore$ required velocities of A, B, C, D, are 1, 2, 4, 8, inches per second respectively.

7. A town is supplied with gas at a stated price per thousand cubic feet whenever the annual consumption and price of coals are certain fixed quantities; but it is agreed, that in any year when they differ from these values, a variation in the former at the rate of m per cent. shall cause an *opposite* variation in the stated price of gas at the rate of n per cent. ($n < m$) and a variation in the latter at the rate of p per cent. shall cause a *like* variation in the stated price of gas at the rate of q per cent. After r years the *average* annual consumption and *average* price of coals have accorded with their assigned values, and yet the town has paid £P more than it would have done, had there been no variation from the assigned values of the annual consumption and price of coals. But if the above-mentioned relative variations in the consumption and price of gas had been p and q per cent., and those of the prices of coals and gas m and n per cent., the town would have paid £Q less. Supposing that in any year, the x^{th}, there is a variation in the consumption and price of coals from their fixed values of x per cent., find the greatest sum the town could pay for the gas it consumed in any given year.

$$\left.\begin{aligned} \text{Let } u &= \text{annual consumption in thousand cubic feet,} \\ v &= \text{price (in pounds) per ton of coals,} \\ y &= \text{price (in pounds) per thousand cubic feet of gas,} \end{aligned}\right\}$$

originally fixed upon,

then in the x^{th} year the consumption $= u\left(1+\dfrac{x}{100}\right)$, where x may be positive or negative;

also price of coals in x^{th} year $= v\left(1+\dfrac{x_1}{100}\right)$, where (to avoid confusion in algebraical signs) we use x_1 instead of x, and x_1 may be positive or negative.

Now in x^{th} year $\dfrac{n}{m}x =$ per centage variation in price of gas arising from variation in annual consumption,

also in x^{th} year $\dfrac{q}{p}x_1 =$ per centage variation in price of gas arising from variation in price of coals;

$\therefore$ since variations in consumption and price of gas are *opposite*, and those of price of coals and gas are *like*, the price per thousand cubic feet of gas in x^{th} year

$$= y\left(1-\frac{n}{m}\cdot\frac{x}{100}+\frac{q}{p}\cdot\frac{x_1}{100}\right);$$

$\therefore$ whole cost of gas in x^{th} year

$$=yu\left(1+\frac{x}{100}\right)\left(1-\frac{n}{m}\cdot\frac{x}{100}+\frac{q}{p}\cdot\frac{x_1}{100}\right);$$

$\therefore$ whole cost of gas in r years

$$=yu\Sigma\left\{\left(1+\frac{x}{100}\right)\left(1-\frac{n}{m}\cdot\frac{x}{100}+\frac{q}{p}\cdot\frac{x_1}{100}\right)\right\},$$

$$=yu\left\{r+\frac{1}{100}\left(1-\frac{n}{m}\right)\Sigma(x)+\frac{q}{p}\cdot\frac{1}{100}\Sigma(x_1)-\frac{1}{(100)^2}\cdot\frac{n}{m}\Sigma(x^2)\right.$$

$$\left.+\frac{1}{(100)^2}\cdot\frac{q}{p}\Sigma(xx_1)\right\}\ldots\ldots(a).$$

Now since the *average* price of coals and *average* consumption have been the same as their fixed values,

$$\therefore\ \Sigma(x)=0,\quad \Sigma(x_1)=0;$$

$\therefore$ (a) becomes

$$yu\left\{r-\frac{n}{m}\cdot\frac{1}{(100)^2}(1^2+2^2+3^2+\ldots\ldots r^2)+\frac{1}{(100)^2}\cdot\frac{q}{p}\Sigma(xx_1)\right\},$$

and this, by the question,

$$=ryu+P.$$

Now for $1^2+2^2+3^2+\ldots\ldots r^2$ write S,

$$\text{then } yu\left\{\frac{q}{p}\Sigma(xx_1)-\frac{n}{m}S\right\}=(100)^2P.$$

Similarly, in supposed case,

$$yu\left\{\frac{n}{m}\Sigma(xx_1)-\frac{q}{p}S\right\}=-(100)^2Q.$$

$$\therefore\ yu=\frac{(100)^2}{S}\cdot\frac{P\frac{n}{m}+Q\frac{q}{p}}{\frac{q^2}{p^2}-\frac{n^2}{m^2}}.$$

Hence the cost of gas in a given year, the t^{th},

$$=\frac{(100)^2}{S}\cdot\frac{P\frac{n}{m}+Q\frac{q}{p}}{\frac{q^2}{p^2}-\frac{n^2}{m^2}}\left(1\pm\frac{t}{100}\right)\cdot\left\{1\mp\frac{n}{m}\cdot\frac{t}{100}\pm\frac{q}{p}\cdot\frac{t}{100}\right\}.$$

There are 4 cases to be considered,

1st, when consumption and price of coals both increase,

2nd, when consumption increases, and price of coals decreases,

3rd, ... decreases, ... increases,

4th, ... decreases, ... decreases,

Now evidently cost of gas in case (1) > cost in case (2); also cost in (3) > cost in (4); also cost in (1) > or < cost in (3), according as

$$\left(1+\frac{t}{100}\right)\left(1-\frac{n}{m}\cdot\frac{t}{100}+\frac{q}{p}\cdot\frac{t}{100}\right)$$

$$> \text{or} < \left(1-\frac{t}{100}\right)\left(1+\frac{n}{m}\cdot\frac{t}{100}+\frac{q}{p}\cdot\frac{t}{100}\right),$$

$$\text{that is, as } 1+\frac{t}{100}\cdot\frac{q}{p} > \text{or} < \frac{n}{m};$$

but since $n < m$, the left hand side is the greater;

$\therefore$ cost of gas in case (1) is the greatest;

$\therefore$ greatest cost of gas in t^{th} year

$$= \frac{100^2}{S}\cdot\frac{P.\frac{n}{m}+Q.\frac{q}{p}}{\frac{q^2}{p^2}-\frac{n^2}{m^2}}\cdot\left(1+\frac{t}{100}\right)\cdot\left\{1-\left(\frac{n}{m}-\frac{q}{p}\right)\frac{t}{100}\right\}.$$

VII.

1. $\dfrac{x+1}{x-1}+\dfrac{x-2}{x+2}+\dfrac{x-3}{x+3}+\dfrac{x+4}{x-4}=4$; find x.

$$\frac{x+1}{x-1}-1+\frac{x-2}{x+2}-1+\frac{x-3}{x+3}-1+\frac{x+4}{x-4}-1=0,$$

$$\frac{2}{x-1}-\frac{4}{x+2}-\frac{6}{x+3}+\frac{8}{x-4}=0,$$

$$\frac{4}{x-4}-\frac{3}{x+3}=\frac{2}{x+2}-\frac{1}{x-1},$$

$$\therefore \frac{x+24}{x^2-x-12}=\frac{x-4}{x^2+x-2};$$

$$\therefore\ x\{x^2+x-2-(x^2-x-12)\}=-24(x^2+x-2)-4(x^2-x-12),$$

$$2x^2+10x=96-20x-28x^2,$$

$$30x^2+30x=96,$$

$$x^2+x+\frac{1}{4}=\frac{16}{5}+\frac{1}{4}=\frac{69}{20};$$

$$\therefore\ x=-\frac{1}{2}\left(1\pm\sqrt{\frac{69}{5}}\right).$$

. $(a+x)^8+(a^2+x^2)^4=c^4x^4$; find x.

$$\left(\frac{a^2+x^2}{ax}+2\right)^4+\left(\frac{a^2+x^2}{ax}\right)^4=\frac{c^4}{a^4},$$

for $\dfrac{a^2+x^2}{ax}$ write $y-1$,

then $(y+1)^4+(y-1)^4=\dfrac{c^4}{a^4}$;

or $y^4+6y^2+1=\dfrac{c^4}{2a^4}$;

$$\therefore\ y^2+3=\pm\sqrt{\frac{c^4+16a^4}{2a^4}},$$

$$\therefore\ y=\left\{\sqrt{\frac{c^4+16a^4}{2a^4}}-3\right\}^{\frac{1}{2}}=m, \text{ suppose.}$$

Hence, from (a), $\dfrac{x}{a}+\dfrac{a}{x}=m-1$;

$$\therefore\ \frac{x}{a}-\frac{a}{x}=\left\{(m-1)^2-4\right\}^{\frac{1}{2}}=\pm(m^2-2m-3)^{\frac{1}{2}}.$$

Adding, $\dfrac{2x}{a}=m-1\pm\sqrt{m^2-2m-3}$;

$$\therefore\ x=\frac{a}{2}\left(m-1\pm\sqrt{(m+1)(m-3)}\right), \text{ where } m=\left(\sqrt{8+\frac{c^4}{2a^4}}-3\right)^{\frac{1}{2}}.$$

3. $(x+1)(x^2+1)(x^6-1)=px^4(x-1)$; find x.

$(x+1)(x^2+1)(x^3+1)(x^2+x+1)=px^4$, (also $x=1$ is a solution,)

or $(x+1)^2(x^2+1)(x^4+x^2+1)=px^4$;

$$\therefore\ \left(x+2+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)=p.$$

Let $x+\dfrac{1}{x}=y$: then $(y+2)y(y^2-1)=p$;

$$y^4+2y^3-y^2-2y=p;$$

$$(y^2+y)^2-2(y^2+y)+1=p+1;$$

$$\therefore\ y^2+y=\pm\sqrt{p+1}+1;$$

$$\therefore\ y=\frac{1}{2}\left\{-1\pm\sqrt{\pm4\sqrt{p+1}+5}\right\}=m, \text{ suppose.}$$

$$\therefore\ x+\frac{1}{x}=m, \text{ and } x-\frac{1}{x}=\pm\sqrt{m^2-4};$$

$$\therefore\ x=\frac{1}{2}(m\pm\sqrt{m^2-4}), \text{ where } m \text{ is given above.}$$

4. $\dfrac{a^3}{x}+3yz=\dfrac{b^3}{y}+3zx=\dfrac{c^3}{z}+3xy=xy+yz+zx$; find x, y, z.

Let each of these four expressions $=\rho$, and let $3xyz=m$;

$$\text{then } m+a^3=\rho x,\ m+b^3=\rho y,\ m+c^3=\rho z;$$

$$\therefore\ (m+a^3)(m+b^3)+(m+a^3)(m+c^3)+(m+b^3)(m+c^3)$$
$$=\rho^2(xy+xz+yz)=\rho^3\ \ldots\ldots\ldots\ldots(\alpha).$$

Also $(m+a^3)(m+b^3)(m+c^3)=\dfrac{\rho^3 m}{3}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$.

From (α) and (β), $3(m+a^3)(m+b^3)(m+c^3)$
$$=m\{(m+a^3)(m+b^3)+(m+a^3)(m+c^3)+(m+b^3)(m+c^3)\},$$

or $3\{m^3+m^2(a^3+b^3+c^3)+m(a^3b^3+a^3c^3+b^3c^3)+a^3b^3c^3\}$
$$=3m^3+2m^2(a^3+b^3+c^3)+m(a^3b^3+a^3c^3+b^3c^3).$$

Hence $m^2(a^3+b^3+c^3)+2m(a^3b^3+a^3c^3+b^3c^3)+3a^3b^3c^3=0$.

Now let $a^3+b^3+c^3=p_1$, $a^3b^3+a^3c^3+b^3c^3=p_2$, $a^3b^3c^3=p_3$;

$$\text{then } m=\frac{(\pm\sqrt{p_2^2-3p_1p_3}-p_2)}{p_1};$$

$$\text{and from } (\alpha),\ \rho^3=3m^2+2mp_1+p_2;$$

$$\therefore\ \frac{x}{m+a^3}=\frac{y}{m+b^3}=\frac{z}{m+c^3}=\frac{1}{\rho}=\frac{1}{\sqrt[3]{3m^2+2mp_1+p_2}},$$

where m_1, p_1, p_2, p_3, are given above.

5. A person starts to walk, at an uniform speed, without stopping, from Cambridge to Madingley and back, at the same time that another starts to walk, at an uniform speed, without stopping, from Madingley to Cambridge and back. They meet a mile and a half from Madingley; and again, an hour after, a mile from Cambridge. Find their rates of walking, and the distance between Cambridge and Madingley.

Let x = distance in miles between Cambridge and Madingley,

then while A walks $x-1\frac{1}{2}$ miles, B walks $1\frac{1}{2}$ miles;

also while A walks $2x-1$ miles, B walks $x+1$ miles;

$$\therefore\ x-1\tfrac{1}{2} : 2x-1 :: 1\tfrac{1}{2} : x+1;$$

$$(2x-3)(x+1)=3(2x-1);$$

$$2x^2-x=6x;\ \therefore\ x=3\tfrac{1}{2}.$$

Also in an hour A walks $1\frac{1}{2}+x-1=4$ miles.

....................B walks $x-1\frac{1}{2}+1=3$ miles.

6. Sovereigns in number equal to four hundred times the number representing the ratio of the weight of pure gold to the weight of dross in a Sovereign contain as much pure gold as 133 Napoleons together with Napoleons in number equal to six hundred times the number representing the ratio of the weight of pure gold to the weight of dross in a Napoleon. Also there is as much pure gold in 400 Sovereigns as in 503 Napoleons; and 11000 Sovereigns weigh as much as 13581 Napoleons. How much of (1) a Sovereign, (2) a Napoleon, is pure gold?

Let x represent ratio of pure gold to dross in a sovereign,

y .... Napoleon,

u = weight of a sovereign,

v = ... Napoleon;

then, by the question, $400x$ sovereigns contain as much pure gold as $133+600y$ Napoleons;

$$\left.\begin{aligned}\therefore\ 400x\frac{ux}{x+1}&=(133+600y)\frac{vy}{y+1}\ldots\ldots\ldots(1),\\ 400\frac{ux}{x+1}&=503\frac{vy}{y+1}\ldots\ldots\ldots(2),\\ 11000u&=13581v\ \ldots\ldots\ldots(3).\end{aligned}\right\}$$

From (1) and (2), $503x=133+600y$........................(α),

.... (2) and (3), $55\dfrac{x+1}{x}=54\dfrac{y+1}{y}$........................(β);

$$\therefore\ 55.\frac{636+600y}{133+600y}=54\frac{y+1}{y},$$

$$55y(106+100y)=9(y+1)(133+600y),$$

$$5500y^2+5830y=9(600y^2+733y+133);$$

$$100y^2-767y=1197,$$

$$y^2-\frac{767y}{100}+\left(\frac{767}{200}\right)^2=\left(\frac{1033}{200}\right)^2;$$

$$\therefore\ y-\frac{767}{200}=\pm\frac{1033}{200};$$

using positive sign $y=9$, or ratio required is $9:1$;

$\therefore$ from (α), $503x=133+5400=5533$;

$\therefore$ $x=11$, and the ratio required is $11:1$;

$\therefore$ $\frac{11}{12}$ths of a sovereign, and $\frac{9}{10}$ths of a Napoleon, is pure gold.

7. Suppose that each of the University Presses at Cambridge and Oxford has a fixed demand, every new year's day, for 5000 copies of an edition of the Bible containing 30 sheets, the price of the paper for which is 12*s.* 6*d.* per ream of 500 sheets, the expense of setting up the type £140, and the cost of press-work 5*s.* for every thousand sheets printed. The custom is, at Oxford to keep the type continually standing, at Cambridge to take it down as soon as the printing of what is thought proper at any particular time is completed: in consequence it is necessary to purchase (suppose) twenty times as much type at Oxford as at Cambridge. Assuming that at Oxford the 5000 copies are always printed just before they are wanted, and that at Cambridge a supply is printed at one time for the most advantageous possible number of years (simple interest being reckoned at the rate of 5 per cent. per annum on all money laid out in the material and workmanship of any stock kept on hand one or more years), the capital required for the above purpose at Oxford is £417. 10*s.* less than the *average* capital required for the same purpose at Cambridge. What is the cost of the type for this edition of the Bible at Oxford and at Cambridge?

Cost of 500 sheets is 12*s.* 6*d.*;

$\therefore$ cost of 30 .... 9*d.*;

press work for 1000 sheets is 5*s.*;

$\therefore$ .... 30 .... is 1·8*d.*;

$\therefore$ cost of paper and press-work for one copy of Bible = 9*d.* + 1·8*d.*

= 10·8*d.*

= ·9*s.*

$\therefore$ .... for 5000 copies $=\text{£}\,\frac{\cdot 9\times 5000}{20}$

= £225.

At Cambridge let a supply for n years be printed at once. Then as simple interest is reckoned at 5 per cent., at the n^{th} New Year's day, the money value of the 5000 copies is $£225+£225.\frac{n-1}{20}$.

And this must not be greater than the expense of bringing out an edition at the time, i. e. £225+£140, the latter being the cost of composition.

$$\therefore\ 225.\frac{n-1}{20} \text{ is not} > 140;\ \therefore\ n-1 \text{ is not} > \frac{20\times 140}{225},$$

$$\therefore\ n \text{ not} > 1+\frac{112}{9}, \text{ i. e. not} > 13;$$

$\therefore$ a supply for 13 years is the most advantageous.

Let x = cost of type at Cambridge,

$20x$ = ... Oxford;

$\therefore$ money invested at Cambridge in type $= £x$,

.... composition $=140$,

.... paper and press-work $=225\times 13$.

Let $x+140+225\times 13=13y$.

Then during the

1st year there is a dead stock whose prime cost $=12y$,

2nd.. $=11y$,

3rd.. $=10y$,

&c. &c.

12th.. $=y$;

the average of which is $\frac{y}{12}(12+11+10+\ldots\ldots\ldots+1)=\frac{13y}{2}$;

$$\therefore \text{ average capital at Cambridge} = \frac{x+140+225\times 13}{2},$$

and the capital at Oxford $=20x+140$.

$$\therefore \text{ by question, } 417\tfrac{1}{2}=\tfrac{1}{2}(x+140+225\times 13)-(20x+140),$$
$$=\tfrac{1}{2}(2785-39x);$$

$$\left.\begin{array}{l}\therefore\ x=£50=\text{the cost of type at Cambridge,}\\ \text{and } £1000=\ldots\ldots\ldots\ldots\ldots\ldots\text{Oxford.}\end{array}\right\}$$

VIII.

1. $\sqrt{\dfrac{x^2-2x+3}{x^2+2x+4}}+\sqrt{\dfrac{x^2+2x+4}{x^2-2x+3}}=2\frac{1}{2}$; find x.

Let $\dfrac{x^2-2x+3}{x^2+2x+4}=y^2$, then $y+\dfrac{1}{y}=2\frac{1}{2}$;

$$\therefore y^2-\frac{5}{2}y+\frac{25}{16}=\frac{25}{16}-1=\frac{9}{16};$$

$$y-\frac{5}{4}=\pm\frac{3}{4};\quad \therefore y=2, \text{ or } \frac{1}{2};$$

$$\therefore \frac{x^2-2x+3}{x^2+2x+4}=4, \text{ or } \frac{1}{4};$$

the former value gives $x=\frac{1}{3}(-5\pm\sqrt{-14})$; the latter gives $x=2$, or $\frac{4}{3}$.

2. $\left.\begin{array}{r} x^3+3axy+y^3=b^3 \\ x+y=c \end{array}\right\}\begin{array}{l}\ldots(1), \\ \ldots(2);\end{array}$ find x and y.

Cubing (2), we have $x^3+3xyc+y^3=c^3$;

$$\therefore 3xy(a-c)=b^3-c^3;$$

$$\therefore 4xy=\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\ldots.(3).$$

Squaring (2), and subtracting (3) from it,

$$x-y=\pm\left(c^2-\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\right)^{\frac{1}{2}};$$

$$\therefore x=\tfrac{1}{2}\left\{c\pm\left(c^2-\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\right)\right\},$$

$$\text{and } y=\tfrac{1}{2}\left\{c\mp\left(c^2-\frac{4}{3}\cdot\frac{b^3-c^3}{a-c}\right)^{\frac{1}{2}}\right\}.$$

3. $x^{\frac{1}{3}}+\dfrac{1}{x^{\frac{1}{3}}}=\sqrt{\dfrac{1}{3}\cdot\dfrac{x^{\frac{3}{2}}+\dfrac{1}{x^{\frac{3}{2}}}}{x^{\frac{1}{2}}+\dfrac{1}{x^{\frac{1}{2}}}}}$; find x.

$$x^{\frac{1}{3}}+\frac{1}{x^{\frac{1}{3}}}=\sqrt{\frac{1}{3}\left(x-1+\frac{1}{x}\right)};$$

$$\text{let } x^{\frac{1}{3}}+\frac{1}{x^{\frac{1}{3}}}=y, \text{ then } x+\frac{1}{x}=y^3-3y,$$

$$\therefore\ y^2=\frac{1}{3}(y^3-3y-1),$$

$$y^3-3y^2-3y-1=0;$$

$$2y^3=(y+1)^3,$$

$$\sqrt[3]{2}\,.\,y=y+1;\ \therefore\ y=\frac{1}{\sqrt[3]{2}-1}=a, \text{ suppose};$$

$$\therefore\ x+\frac{1}{x}=a^3-3a;\ \therefore\ x=\frac{a^3-3a}{2}\left\{1\pm\sqrt{1-\frac{4}{(a^3-3a)^2}}\right\}.$$

4. $$\left.\begin{array}{r} x^2+y^2+z^2=38,\ldots\ldots\ldots\ldots\ldots\ldots(1)\\ 2x+3y+5z=29,\ldots\ldots\ldots\ldots\ldots\ldots(2)\\ 15x^2+10y^2+6z^2=12xz+12yz+297,\ldots(3)\end{array}\right\};\ \text{find } x,\ y,\ z.$$

From (3), $19(x^2+y^2+z^2)=(2x+3z)^2+(3y+2z)^2+297$;

$\therefore$ $(2x+3z)^2+(3y+2z)^2=19\times38-297=722-297=425.$

For $2x+3z$ write u, and for $3y+2z$ write v, then $u^2+v^2=425$,

and from (2), $u+v=29$;

$$\therefore\ 2uv=841-425=416;$$

$$\therefore\ (u-v)^2=9;\ \therefore\ u-v=\pm3;\ \therefore\ \left.\begin{array}{r}u=16, \text{ or } 13,\\ v=13, \text{ or } 16.\end{array}\right\}\ldots\ldots\ldots(4).$$

Taking first pair of values, $\left.\begin{array}{r}2x+3z=16,\\ 3y+2z=13,\end{array}\right\}\begin{array}{l}\ldots\ldots\ldots(\alpha).\\ \ldots\ldots\ldots(\beta).\end{array}$

Substituting in (1), $\frac{1}{4}(16-3z)^2+\frac{1}{9}(13-2z)^2+z^2=38$,

$$9(16-3z)^2+4(13-2z)^2+36z^2=1368,$$

$$133z-1072z=-1612,$$

$$z^2-\frac{1072}{133}\,.\,z+\left(\frac{536}{133}\right)^2=\left(\frac{536}{133}\right)^2-\frac{1612}{133}=\frac{72900}{(133)^2}=\left(\frac{270}{133}\right)^2;$$

$$\therefore\ z-\frac{536}{133}=\pm\frac{270}{133};\ \therefore\ z=\frac{806}{133}, \text{ or } 2;$$

$$\therefore \text{ from } (\alpha) \text{ and } (\beta),\ x=5,\ y=3.$$

Similarly we shall get different values of x, y, z, by taking for u and v the other pair of values in (4).

5. A person who copied a manuscript in a regular manner found that the number of lines copied in the first half hour was less by ten than the square root of the whole number of lines in the manuscript; and that the square of the number of lines copied in the first 49 minutes was equal to twice the number of lines then remaining to be copied. How many lines were there in the manuscript?

Let x^2 = No. of lines in MS.;

$\therefore$ No. of lines copied in first half hour $= x-10$;

$\therefore$ No. .... in 49 minutes $= \frac{49}{30}(x-10)$;

$$\therefore \text{ by question, } \left(\frac{49}{30}\right)^2(x-10)^2 = 2\left\{x^2 - \frac{49}{30}(x-10)\right\},$$

$$\text{or } (49)^2(x^2-20x+100) = 1800x^2 - 60\times49(x-10),$$

$$\text{or } 601x^2 - 20x\times2254 = -100\times49\times43;$$

$$\therefore x^2 - 20x\,.\,\frac{2254}{601} + 100\left(\frac{2254}{601}\right)^2 = \left(\frac{1953}{601}\right)^2\times100;$$

$$\therefore x - 10\,.\,\frac{2254}{601} = \pm\frac{1953}{601}\times10,$$

using positive sign, $x=70$:

$\therefore$ No. of lines in MS. $= 4900$.

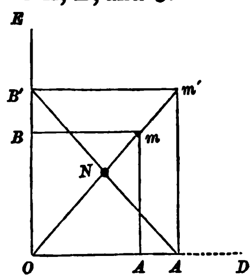
6. A, B, C walk uniformly in three roads, starting at the same moment from their point of intersection. The roads in which A and B walk are at right angles; and the road in which C walks lies somewhere between them. A line joining the positions of A and C at any time is parallel to the road in which B walks; and a line joining the positions of B and C at any time is parallel to the road in which A walks. When C has walked 8 miles more than A, he begins to return, and when he has walked $5\frac{1}{3}$ miles more than B, all three are in the same straight line. Find the *relative* speeds of A, B, and C.

Let x, y, z be the rates in miles per hour of A, B, C.

Now it is clear that A, B, C, walk in the adjacent sides and diagonal of a rectangle, respectively;

and if m be the position of C when he has gone 8 miles more than A,

by dropping perpendiculars on OD and OE from m, we get positions of A and B at that time.



Also suppose C, instead of returning, had gone as far *forward*, as he comes *back again*, before he is in a line with A and B; and let m', A', B' be the simultaneous positions of C, A, B, on this supposition; join $A'B'$ cutting Om' in N, then N is clearly the position of C when he is in a line with A and B.

Now time that C takes to travel $Om = \dfrac{8}{z-x}$;

and time C takes to travel Om and back again to $N = \dfrac{5\frac{1}{3}}{z-y}$;

and since $mN = mm' = \frac{1}{2} ON = \frac{1}{3} Om$; $\therefore\ Om + mN = \frac{4}{3} Om$;

$$\therefore\ \frac{4}{3} \cdot \frac{8}{z-x} = \frac{16}{3} \cdot \frac{1}{z-y}, \quad \therefore\ 2z - 2y = z - x, \quad z = 2y - x.$$

$$\text{But } z^2 = x^2 + y^2;$$

$$\therefore\ 4y^2 - 4yx + x^2 = x^2 + y^2; \quad \therefore\ 3y = 4x, \text{ and } z = \frac{8x}{3} - x = \frac{5x}{3};$$

$\therefore\ x : y : z :: 3 : 4 : 5$, the relative speeds of A, B, C.

7. A farmer has between 270 and 280 quarters of wheat, which he agrees to sell to a miller on the following terms. Not less than 20, and not more than 30, quarters (the exact number being at the option of the farmer) are to be delivered at the beginning of each month of the year, for which the farmer is to be paid according to the market price of wheat at the time of each delivery. The price of wheat at the beginning of June is the same as the price at the beginning of July; and the price increases uniformly a certain number of shillings per quarter each month during the first six months of the year, and decreases uniformly the same number of shillings per quarter each month during the last six months of the year. The farmer, by making the most of his wheat under this contract, receives £7. 8*s*. more for it than he would have done had he delivered the same average quantity each month. If, however, the number of quarters to be delivered each month had been not less than 20, and not greater than 25 (the exact number being at the option of the miller), the miller, by arranging the delivery in the way most advantageous to himself, would have paid £11. 12*s*. less than he actually paid; and the amount paid in April and September together in the latter case would have been £33. 12*s*. less than the amount paid in May and August together in the former case. How much wheat had the farmer, and what did he receive for it?

Let $270 + x =$ No. of quarters, $y =$ price (in shillings) per quarter at beginning of June and July.

$z =$ monthly increase up to June, and corresponding decrease during latter half of the year;

then price of wheat in Jan. and Dec. $=y-5z$; in Feb. and Nov. $=y-4z$; in March and Oct. $=y-3z$; in April and Sept. $=y-2z$; in May and Aug. $=y-z$; in June and July $=y$.

Now clearly the farmer, in order to make the most possible of his wheat, must deliver as much as he can (consistently with the conditions and limitations of the problem) in the middle months of the year. Hence he must deliver 60 quarters in June and July, $50+x$ in May and Aug., and 20 quarters in each of the other 8 months;

$\therefore$ what he makes of his wheat

$$=40\{(y-5z)+(y-4z)+(y-3z)+(y-2z)\}+(50+x)(y-z)+60y,$$

and this, by question,

$$=\frac{270+x}{6}(6y-15z)+148\ldots\ldots\ldots\ldots(1).$$

In the supposed case the miller would have

50 quarters delivered in Jan. and Dec.

50 .... Feb. and Nov.

50 .... March and Oct.

$40+x$ .... April and Sept.,

and 20 quarters during each of the remaining 4 months;

$\therefore$ cost of wheat

$$=50\{(y-5z)+(y-4z)+(y-3z)\}+(40+x)(y-2z)+40\{(y-z)+y\},$$

and this, by question,

$$=40(4y-14z)+(50+x)(y-z)+60y-232\ldots\ldots(2).$$

Again, $(40+x)(y-2z)=(50+x)(y-z)-672\ldots\ldots\ldots(3)$.

$$\left.\begin{aligned}\text{From (1)},\quad & z\left(65+\frac{3x}{2}\right)=148\\ (2),\quad & z(110+x)=232\end{aligned}\right\};$$

$$\therefore\ 232\left(65+\frac{3x}{2}\right)=148(110+x).$$

$$\left.\begin{aligned}\text{From which } & x=6,\\ \text{and } \therefore\ & z=2.\end{aligned}\right\}$$

From (3), $46(y-4)=56(y-z)-672$;

$$\therefore\ 10y=672+112-184=600;$$

$$\therefore\ y=60;$$

$\therefore$ the farmer had 276 quarters of wheat, and he received for it

$40(240-28)+56\times58+3600$ shillings $=$ £766. 8*s*.

IX.

1. $\dfrac{5}{x-1}+\dfrac{4}{x+2}+\dfrac{21}{x-3}=\dfrac{5}{x+1}+\dfrac{4}{x-2}+\dfrac{21}{x+3}$; find x.

$$\frac{10}{x^2-1}-\frac{16}{x^2-4}+\frac{126}{x^2-9}=0,$$

$$5(x^4-13x^2+36)-8(x^4-10x^2+9)+63(x^4-5x^2+4)=0,$$

$$x^4-5x^2+6=0;$$

$$\therefore\ x^2=2,\ \text{or } 3,$$

$$\text{and}\quad x=\pm\sqrt{2},\ \text{or } \pm\sqrt{3}.$$

2. $\dfrac{x^2}{3}+\dfrac{48}{x^2}=10\left(\dfrac{x}{3}-\dfrac{4}{x}\right)$; find x.

$$\frac{x^2}{9}+\frac{16}{x^2}=\frac{10}{3}\left(\frac{x}{3}-\frac{4}{x}\right),$$

$$\left(\frac{x}{3}-\frac{4}{x}\right)^2=\frac{10}{3}\left(\frac{x}{3}-\frac{4}{x}\right)-\frac{8}{3};$$

$$\therefore\ \frac{x}{3}-\frac{4}{x}=\frac{4}{3},\ \text{or } 2.$$

$$\therefore\ x=6,\ \text{or } -2,\ \text{or } 3\pm\sqrt{21}.$$

3. $\dfrac{(x+a+b)^5+(x+c+d)^5}{(x+a+c)^5+(x+b+d)^5}=\left(\dfrac{a+b-c-d}{a-b+c-d}\right)^2$; find x.

$$\text{Let}\ \left.\begin{aligned} a+b&=\alpha+\beta\\ c+d&=\alpha-\beta\\ a+c&=\alpha'+\beta'\\ b+d&=\alpha'-\beta'\end{aligned}\right\};\qquad \therefore\ \left\{\begin{aligned}\alpha&=\tfrac{1}{2}(a+b+c+d),\\ \beta&=\tfrac{1}{2}(a+b-c-d),\\ \alpha'&=\tfrac{1}{2}(a+b+c+d)=\alpha,\\ \beta'&=\tfrac{1}{2}(a-b+c-d).\end{aligned}\right.$$

Let $y=x+\alpha$, then $\dfrac{(y+\beta)^5+(y-\beta)^5}{(y+\beta')^5+(y-\beta')^5}=\dfrac{\beta^2}{\beta'^2}$;

$$\therefore\ \frac{y^5+10y^3\beta^2+5y\beta^4}{y^5+10y^3\beta'^2+5y\beta'^4}=\frac{\beta^2}{\beta'^2};$$

$$\therefore\ y=0,\ \text{and}\ x=-\tfrac{1}{2}(a+b+c+d),$$

$$\text{or}\ \frac{y^4+10y^2\beta^2+5\beta^4}{y^4+10y^2\beta'^2+5\beta'^4}=\frac{\beta^2}{\beta'^2};$$

$$\therefore\ (\beta''^2-\beta^2)y^4+5(\beta''^2\beta^4-\beta^2\beta'^4)=0;$$

$$\therefore\ y^4=5\beta^2\beta''^2;$$

$$y=\pm\frac{1}{2}\sqrt[4]{5(a+b-c-d)^2(a-b+c-d)^2};$$

$$\therefore\ x=-\frac{1}{2}(a+b+c+d)\pm\frac{1}{2}\sqrt[4]{5(a+b-c-d)^2(a-b+c-d)^2};$$

4. $\left.\begin{aligned}\frac{(ac+1)(x^2+1)}{x+1}&=\frac{(a^2+1)(xy+1)}{y+1}\\ \frac{(ac+1)(y^2+1)}{y+1}&=\frac{(c^2+1)(xy+1)}{x+1}\end{aligned}\right\}$; find x and y.

$$\frac{(x^2+1)(y^2+1)}{(xy+1)^2}=\frac{(a^2+1)(c^2+1)}{(ac+1)^2}.$$

Subtract 1, then $\left(\dfrac{x-y}{xy+1}\right)^2=\left(\dfrac{a-c}{ac+1}\right)^2$;

$$\therefore\ \frac{x-y}{xy+1}=\frac{a-c}{ac+1},\ \text{or}\ \frac{c-a}{ac+1}.$$

Taking the 1st case, and writing m for $\dfrac{a-c}{ac+1}$,

$$y(1+mx)=x-m;$$

$$\therefore\ \frac{xy+1}{y+1}=\frac{x^2+1}{1+mx+x-m};$$

$\therefore$ from 1st of given equations,

$$\frac{x^2+1}{x+1}=\frac{a^2+1}{ac+1}\cdot\frac{x^2+1}{1+mx+x-m};$$

$$\therefore\ x^2+1=0;\quad\therefore\ x=y=\pm\sqrt{-1},$$

$$\text{or}\ \frac{1}{x+1}=\frac{a^2+1}{(ac+1)(1+mx+x-m)};$$

$$\therefore\ x=\frac{1+a}{1-a},\ \text{and}\ y=\frac{1+c}{1-c}.$$

Taking the 2nd case we have

$$x=\frac{1-a}{1+a},\ \text{and}\ y=\frac{1-c}{1+c}.$$

5. Each of three cubical vessels A, B, C, whose capacities are as 1 : 8 : 27, respectively, is partially filled with water, the quantities of water in them being as 1 : 2 : 3, respectively. So much water is now poured from A into B, and so much from B

into C, as to make the depth of water the same in each vessel. After this, $128\frac{4}{7}$ cubic feet of water is poured from C into B, and then so much from B into A, as to leave the depth of water in A twice as great as the depth of water in B. The quantity of water in A is now less by 100 cubic feet than it was originally. How much water did each of the vessels originally contain?

Let c, $2c$, $3c$, be the sides of the vessels in feet,

x the height of water in A;

then quantity of water in $A = c^2x$,

$$\ldots\ldots\ldots\ldots\ldots\ldots \; B = 2c^2x,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \; C = 3c^2x;$$

$$\therefore \text{ height of water in } B = \frac{2c^2x}{4c^2} = \frac{x}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \; C = \frac{3c^2x}{9c^2} = \frac{x}{3}.$$

Let y be the common depth of the 1st pouring.

Then $(c^2+4c^2+9c^2)y =$ whole quantity of water $= 6c^2x$;

$$\therefore \; y = \frac{3}{7}x.$$

$128\frac{4}{7}$ cubic feet of water being now poured into B, the height of water in $B = y + \dfrac{128\frac{4}{7}}{4c^2} = \dfrac{3x}{7} + \dfrac{225}{7c^2}$.

The whole quantity of water in A and B;

$$= (c^2+4c^2)y + 128\tfrac{4}{7},$$

$$= \frac{15c^2x}{7} + \frac{900}{7}.$$

After the next pouring, let $2z$, z, be the depths in A and B;

$$\text{then } c^2.2z + 4c^2.z = \frac{15c^2x}{7} + \frac{900}{7};$$

$$\therefore \; z = \frac{5x}{14} + \frac{150}{7c^2}.$$

Also, by question,

$$c^2.2z = c^2x - 100;$$

$$\therefore \; \frac{5c^2x + 300}{7} = c^2x - 100;$$

$$\therefore \; c^2x = 500;$$

$\therefore$ the quantities of water required are 500, 1000, 1500, cubic feet.

6. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article; thereby gaining at the rate of 11 per cent. more on his outlay than he would gain were the balance *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Determine the legitimate gain per cent. on the article.

Let w and w' be the apparent weights of the article when bought and sold, respectively;

w being less, and w' greater, than the true weight.

Let p = prime cost of an unit of weight,

x = the legitimate gain per cent.,

then the article costs pw, and is sold for $pw'\left(1+\frac{x}{100}\right)$;

$\therefore$ by question,

$$pw'\left(1+\frac{x}{100}\right)=pw\left(1+\frac{x+11}{100}\right).$$

Again, in the supposed case, the article costs pw', and is sold for $pw\left(1+\frac{x}{100}\right)$;

$$\therefore\ pw\left(1+\frac{x}{100}\right)=pw';$$

$$\therefore\ \left(1+\frac{x}{100}\right)^2=1+\frac{x+11}{100},$$

which gives $x=10$,

or the legitimate gain is 10 per cent.

7. A metallic lump m_1 is compounded of the metals a and b, another lump m_2 of the metals b and c. The ratio of the weight of a in m_1 to that of c in m_2 is three times the ratio of the weights of b in m_1, m_2, respectively; and three times the weight of m_1 is ten times the weight of m_2. In two other lumps μ_1, μ_2, of the same volumes and compounded of the same metals as m_1, m_2 respectively, the ratio of the weights of a and b in μ_1 is equal to the ratio of the weights of b and c in m_2; the ratio of the weights of b and c in μ_2 is equal to the ratio of the weights of a and b in m_1; and five times the weight of μ_1 is eighteen times the weight of μ_2. Having given that the weights of equal volumes of a, b, c, are as 3 : 2 : 1 respectively, determine the proportions in which each of the lumps is compounded.

Let V, v, be the bulks of a, b, in m_1,
V_1, v_1 .... b, c, ... m_2,
U, u .... a, b, ... μ_1,
U_1, u_1 .... b, c, ... μ_2,

and 3λ, 2λ, λ, the weights of unit of bulk of a, b, c;

$\therefore$ ratio of weight of a in m_1 to c in m_2 is $\dfrac{3V}{v_1}$,

.... b in m_1 to b in m_2 is $\dfrac{v}{V_1}$;

$$\therefore \text{ by question, } \frac{V}{v_1} = \frac{v}{V_1} \text{ .... (1).}$$

Again, $\because$ 3 times weight of $m_1 = 10$ times weight of m_2;

$$\therefore\ 3(3V+2v) = 10(2V_1+v_1) \text{ ....(2),}$$

$$\text{and } 5(3U+2u) = 18(2U_1+u_1) \text{ ....(3).}$$

Also as the lumps are of same volume as original lumps,

$$V+v = U+u \text{ ....(4),}$$

$$V_1+v_1 = U_1+u_1 \text{ ....(5).}$$

Also ratio of weights of a and b in μ_1 is $\dfrac{3U}{2u}$, and of b and c in μ_2 is $\dfrac{2U_1}{u_1}$;

$$\therefore \text{ by question, } \frac{3U}{2u} = \frac{2V_1}{v_1} \text{ ....(6),}$$

$$\frac{2U_1}{u_2} = \frac{3V}{2v} \text{ ....(7).}$$

From (1), $\dfrac{V}{v} = \dfrac{v_1}{V_1} = x$, suppose.

From (2), $3v(3x+2) = 10v_1\left(\dfrac{2}{x}+1\right)$....................($\alpha$).

Taking into account (6) and (7), we have from (3) by transposing,

$$18u_1\left(\frac{3}{2}x+1\right) = 5u\left(\frac{4}{x}+2\right) \text{ ....(}\beta\text{),}$$

from (6) and (4), $\qquad u\left(\dfrac{4}{3x}+1\right) = v(x+1)$.............($\gamma$),

from (7) and (5), $\qquad v_1\left(\dfrac{1}{x}+1\right) = u_1\left(\dfrac{3x}{4}+1\right)$..........($\delta$).

Multiply (α), (β), (γ), (δ) together, and we have

$$9(3x+2)^2=25(x+2)^2,$$

$$3(3x+2)=5(x+2);$$

$$\therefore\ x=1;$$

$$\therefore\ V=v,\quad V_1=v_1,\quad \frac{U}{u}=\frac{4}{3}.\frac{U_1}{u_1}=\frac{3}{4};$$

$\therefore$ m_1 and m_2 are compounded of equal volumes of the metals,

μ_1 is compounded of volumes of a and b in ratio $4:3$,

μ_2 .... b and c ...... $3:4$.

If the proportions of the *weights* be required,

in m_1 the weight of a : weight of b :: $3V$: $2v$ :: $3:2$,

... m_2 .... b : .... c :: $2V_1$: v_1 :: $2:1$,

... μ_1 .... a : .... b :: $3U$: $2u$ :: $2:1$,

... μ_2 .... b : .... c :: $2U_1$: u_1 :: $3:2$.

X.

1. $(x+1)^5+(x-1)^5=19\{(x+1)^3+(x-1)^3\}$; find x.

$$x^5+10x^3+5x=19(x^3+3x);$$

$$\therefore\ x=0,\ \text{or}\ x^4+10x^2+5=19(x^2+3);$$

$$x^4-9x^2=52;$$

$$\therefore\ x^2=13,\ \text{or}\ -4;$$

$$\therefore\ x=\pm\sqrt{13},\ \text{or}\ \pm 2\sqrt{-1}.$$

2. $(x+1)(x+2)(x+3)(x+4)=(x+1)^2+(x+2)^2+(x+3)^2+(x+4)^2$; find x.

For $x+2\frac{1}{2}$ write y; then $\left(y-\frac{3}{2}\right)\left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right)\left(y+\frac{3}{2}\right)$

$$=\left(y-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2,$$

$$\left(y^2-\frac{9}{4}\right)\left(y^2-\frac{1}{4}\right)=2\left\{\left(y^2+\frac{9}{4}\right)+\left(y^2+\frac{1}{4}\right)\right\},$$

$$\therefore\ y^4-\frac{5}{2}y^2+\frac{9}{16}=4y^2+5;$$

$$\therefore\ y^2=\frac{13}{4}\pm\sqrt{15}.$$

$$\therefore\ x=-\frac{5}{2}\pm\sqrt{\frac{13}{4}\pm\sqrt{15}}.$$

3. $\left.\begin{aligned}(x^2+1)(y^2+1)&=10,\\(x+y)(xy-1)&=\ \ 3;\end{aligned}\right\}$ find x and y.

The first equation is, otherwise written,

$$(xy-1)^2+(x+y)^2=10,$$

adding and subtracting twice the second,

$$\left.\begin{aligned}\{(xy-1)+(x+y)\}^2&=16,\\\{(xy-1)-(x+y)\}^2&=\ \ 4,\end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned}(xy-1)+(x+y)&=\pm 4,\\(xy-1)-(x+y)&=\pm 2,\end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned}xy-1&=\pm 3,\ \text{or}\ \pm 1,\\x+y&=\pm 1,\ \text{or}\ \pm 3,\end{aligned}\right\}$$

in each of these pairs the same sign is to be taken;
$\therefore$ we have from the 1st pair,

$$\left.\begin{aligned}x+y&=1,\\xy&=4,\end{aligned}\right\}\ \text{or}\ \left.\begin{aligned}x+y&=-1,\\xy&=-2,\end{aligned}\right\}$$

from the 2nd,

$$\left.\begin{aligned}x+y&=3,\\xy&=2,\end{aligned}\right\}\ \text{or}\ \left.\begin{aligned}x+y&=-3,\\xy&=\ \ 0,\end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned}&x=\frac{1\pm\sqrt{-15}}{2},\ y=\frac{1\mp\sqrt{-15}}{2}\\&x=1,\ \text{or}\ -2,\quad y=-2,\ \text{or}\ 1,\\&x=2,\ \text{or}\ 1,\quad y=1,\ \text{or}\ 2,\\&x=3,\ \text{or}\ 0,\quad y=0,\ \text{or}\ -3.\end{aligned}\right\}$$

4. $\left.\begin{aligned}x^2+y^2-z^2&=(x+y-z)^2+\ 2,\\x^3+y^3-z^3&=(x+y-z)^3+\ 9,\\x^4+y^4-z^4&=(x+y-z)^4+29,\end{aligned}\right\}$ find x, y, z.

Let $x+y-z=u$,

$$\text{then}\ \left.\begin{aligned}x+y&=z+u,\\x^2+y^2&=z^2+u^2+\ 2,\\x^3+y^3&=z^3+u^3+\ 9,\\x^4+y^4&=z^4+u^4+29;\end{aligned}\right\}$$

squaring the 1st of these, subtracting the 2nd, and dividing by 2,

$$xy = zu - 1 \dots\dots\dots\dots\dots\dots (\alpha);$$

cubing the 1st, subtracting the 3rd, and dividing by 3,

$$xy(x+y) = zu(z+u) - 3;$$

$$\therefore\ (zu - xy)(x+y) = 3;$$

$$\therefore \text{ by } (\alpha),\ x+y = 3 = z+u \dots\dots\dots (\beta).$$

Also $x^4+y^4+4xy(x^2+y^2)+6x^2y^2 = z^4+u^4+4zu(z^2+u^2)+6z^2u^2$;

$\therefore$ from the 4th of the above equations,

$$29+4xy(x^2+y^2)+6x^2y^2 = 4zu(z^2+u^2)+6z^2u^2;$$

$$\therefore\ 29+4xy(x+y)^2-2x^2y^2 = 4zu(z+u)^2-2z^2u^2;$$

$$\therefore\ 29 = 4(zu-xy).9-2(z^2u^2-x^2y^2), \text{ from } (\beta),$$

$$= 36-2(zu+xy), \text{ from } (\alpha);$$

$$\therefore\ zu+xy = \frac{7}{2},$$

$$\text{also } zu-xy = 1;$$

$$\therefore \left.\begin{array}{l} zu = \frac{9}{4}, \\ z+u = 3. \end{array}\right\} \quad \left\{\begin{array}{l} xy = \frac{5}{4}, \\ x+y = 3; \end{array}\right.$$

$$\therefore\ z-u = 0, \quad x-y = \pm 2,$$

$$z = u = \frac{3}{2}; \quad x = \frac{5}{2}, \text{ or } \frac{1}{2}; \quad y = \frac{1}{2}, \text{ or } \frac{5}{2}.$$

5. Alfred, Edward, and Herbert come each with his pail to a well; when a question arises about the quantity of water in the well: but none of them knowing how much his pail will hold they cannot settle the dispute. Luckily Mary comes up with a pint measure, by aid of which they discover that Alfred's pail holds half a gallon more than Edward's and a gallon more than Herbert's: but before the precise content of any pail is found out an accident happens and Mary's measure is broken. They are now however in a position to ascertain the quantity of water in the well: for they find that it fills each pail an exact number of times; and that the number of times it fills Edward's is greater by eight than the number of times it fills Alfred's, and less by forty than the number of times it fills Herbert's. How much water was there in the well?

$$\left.\begin{array}{rll} \text{Let } x = & \text{content of Herbert's pail} & \\ x+2 = & \ldots\ldots\ldots\ldots \text{ Edward's} & \ldots \\ x+4 = & \ldots\ldots\ldots\ldots \text{ Alfred's} & \ldots \\ y = & \ldots\ldots\ldots\ldots \text{ well} & \end{array}\right\} \text{in quarts;}$$

$$\therefore\ \frac{y}{x+2} = \frac{y}{x+4} + 8 = \frac{y}{x} - 40.$$

$$\therefore\ \frac{2y}{(x+2)(x+4)} = 8, \text{ and } \frac{2y}{x(x+2)} = 40;$$

$$\therefore\ \frac{x+4}{x} = 5;$$

$$\therefore\ x = 1,$$

and $y = 20x(x+2) = 60$ quarts $= 15$ gallons.

6. Seven-ninths of the stronger of two glasses of wine and water of equal size is mixed with two-ninths of the weaker, and the remainder of the weaker with the remainder of the stronger. The stronger of these two new glasses is a certain number of times stronger than the weaker; and the stronger of the two original glasses was twice the same number of times stronger than the weaker. Compare the strengths of the two original glasses; the strength of a glass of wine and water being defined to be the ratio of the quantity of wine to the quantity of the whole mixture in the glass.

Let x, y, be the quantity of wine in the two glasses; then as they are of the same size, these represent the relative strengths.

The quantity of wine in the mixtures

$$= \frac{1}{9}.(7x+2y), \text{ and } \frac{1}{9}.(7y+2x);$$

$\therefore$ for the same reason as before, the relative strengths of these

$$= \frac{7x+2y}{7y+2x} \text{ which} = \tfrac{1}{2}.\left(\frac{x}{y}\right), \text{ by the question.}$$

Let $\dfrac{x}{y} = u$, then we have $2u^2 - 7u = 4$;

$$\therefore\ u = 4,$$

i. e. the strengths of the original glasses are as $4 : 1$.

7. Two Tyrolese Jäger agree to shoot at a mark on the following terms; each is to shoot an even number of times fixed for each beforehand, and for every time that either hits he is to receive from the other a number of kreuzers equal to the whole

number of times that he misses. They have two matches without varying the conditions. In the first match, the second Jager misses as often as the first hits in the second match, and the first Jager misses twice as often as the second hits in the second match; and the second Jager has to pay to the first a balance of 4 kreuzers. In the second match, each hits exactly the number of times most favourable to him, and the second has to pay to the first a balance of 36 kreuzers. How many times did each hit and miss in each match?

In the second match it may be easily seen that each must miss as often as he hits.

Let then x be the number of times the 1st Jager hits and misses in the 2nd match,

y the number of times the 2nd Jager hits and misses in the 2nd match;

$\therefore$ in 1st match, 1st Jager misses $2y$ times, and hits $2x-2y$ times;

2nd Jager misses x times, and hits $2y-x$ times;

$\therefore$ by question,

$$\left.\begin{aligned}(2x-2y).2y-(2y-x).x=4, \text{ or } 2xy-4y^2+x^2 &= 4,\\ \text{and } x^2-y^2 &= 36.\end{aligned}\right\}$$

Let $x=u+v$, $y=u-v$;

$$\therefore\ 2(u^2-v^2)-4(u^2-2uv+v^2)+u^2+2uv+v^2=4;$$

$$\therefore\ 10uv-u^2-5v^2=4;$$

$$\text{also } 4uv=36, \text{ or } uv=9.$$

Combining these two, we have $u^2+5v^2=86$;

$$\therefore\ u^2+5v^2\pm 2uv\sqrt{5}=86\pm 18\sqrt{5}:$$

$$\therefore\ u\pm\sqrt{5}.v=9\pm\sqrt{5};$$

$$\text{i.e. } u=9,\ v=1;$$

$$\therefore\ x=10,\ y=8;$$

$\therefore$ in 1st match, 1st Jager hits 4 times, and misses 16,
2nd Jager hits 6 times, and misses 10;

in 2nd match, 1st Jager hits 10 times, and misses 10,
2nd Jager hits 8 times, and misses 8.

XI.

1. $\left(\frac{x+a}{x+b}\right)^2+\left(\frac{x-a}{x-b}\right)^2=\left(\frac{a}{b}+\frac{b}{a}\right)\cdot\frac{x^2-a^2}{x^2-b^2}$; find x.

For $\frac{x+a}{x+b}$ write u, and for $\frac{x-a}{x-b}$ write v, then

$$u^2+v^2=\left(\frac{a}{b}+\frac{b}{a}\right)uv,$$

$$u^2-\left(\frac{a}{b}+\frac{b}{a}\right)uv+\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}\right)^2v^2=\left\{\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}\right)^2-1\right\}v^2,$$

$$u-\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)v=\pm\frac{1}{2}\left(\frac{a}{b}-\frac{b}{a}\right)v,$$

$$\therefore\ \frac{u}{v}=\frac{a}{b},\ \text{or}\ \frac{b}{a},$$

$$\text{or}\ \frac{\overline{x+a}.\overline{x+b}}{\overline{x-a}.\overline{x-b}}=\frac{a}{b},\ \text{or}\ \frac{b}{a},$$

$$\frac{x^2-ab}{x(a-b)}=\frac{a+b}{a-b},\ \text{or}\ \frac{a+b}{b-a},$$

$$x^2-ab=\pm(a+b)x,$$

$$\therefore\ x=\tfrac{1}{2}\{\pm(a+b)\pm\sqrt{a^2+6ab+b^2}\}.$$

2. $\left.\begin{array}{l} mx^2+2ny=m,\ldots\ldots(1) \\ ny^2+2mx=n,\ldots\ldots(2)\end{array}\right\}$ find x and y.

$$(1)\times m-(2)\times n\ \text{gives}\ (mx-n)^2=(ny-m)^2,$$

$$\therefore\ mx-n=\pm(ny-m)\ldots\ldots(a).$$

$$=\pm\left\{\frac{m(1-x^2)}{2}-m\right\},$$

$$\therefore\ 2mx-2n=\mp m(1+x^2),$$

$$(x+1)^2=\frac{2n}{m},\ \text{or}\ (x-1)^2=-\frac{2n}{m},$$

$$\therefore\ x=-1\pm\sqrt{\frac{2n}{m}},\ \text{or}\ 1\pm\sqrt{-\frac{2n}{m}}.$$

From (a), $n(y+1)=m(x+1)$, or $n(y-1)=m(1-x)$,

$$\therefore\ y=-1\pm\sqrt{\frac{2m}{n}},\text{ or } 1\mp\sqrt{-\frac{2m}{n}}.$$

3. $$\left.\begin{array}{c}\dfrac{x}{a+y+z}=\dfrac{y}{b+x+z}=\dfrac{z}{c+x+y},\\ (x+y+z)^2=ax+by+cz,\end{array}\right\}\text{ find } x,\ y, \text{ and } z.$$

For $x+y+z$ write u,

$$\text{then } \frac{x}{a+u-x}=\frac{y}{b+u-y}=\frac{z}{c+u-z},$$

$$\therefore\ \frac{x}{a+u}=\frac{y}{b+u}=\frac{z}{c+u},$$

$$\text{and } \therefore \text{ each fraction}=\frac{u}{a+b+c+3u},\ \text{(Art. 195*)}.$$

$$\text{Also each fraction}=\frac{ax+by+cz}{a^2+b^2+c^2+(a+b+c)u},\ \ldots\ldots,$$

$$=\frac{u^2}{a^2+b^2+c^2+(a+b+c)u}.$$

Hence, equating these two values of the fraction,

$$u=\sqrt{\frac{a^2+b^2+c^2}{3}};$$

$$\therefore\ \frac{x}{a+u}=\frac{y}{b+u}=\frac{z}{c+u}=\frac{\sqrt{\dfrac{a^2+b^2+c^2}{3}}}{a+b+c+\sqrt{3(a^2+b^2+c^2)}},$$

$$\therefore\ x=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{a^2+b^2+c^2}}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}\cdot\left\{a+\sqrt{\frac{a^2+b^2+c^2}{3}}\right\},$$

$$=\frac{1}{3}\cdot\frac{a^2+b^2+c^2+a\sqrt{3(a^2+b^2+c^2)}}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}.$$

Similarly y and z.

4. $$\left.\begin{array}{r}a(x+b)^3-b(x+a)^3=b(y+b)^3-a(y+a)^3,\ldots(1)\\ x^2-2y(a+b)=y^2+a^2+b^2,\ \ldots\ldots\ldots\ (2)\end{array}\right\}\text{ find } x \text{ and } y.$$

$$\text{From (1), } \frac{(x+a)^3+(y+b)^3}{(x+b)^3+(y+a)^3}=\frac{a}{b},$$

$$\text{Let } x+\frac{a+b}{2}=u, \quad y+\frac{a+b}{2}=v, \quad \frac{a-b}{2}=\alpha,$$

$$\text{then } \frac{(u+\alpha)^3+(v-\alpha)^3}{(u-\alpha)^3+(v+\alpha)^3}=\frac{a}{b},$$

$$\frac{u^3+v^3+3\alpha^2(u+v)}{3\alpha(u^2-v^2)}=\frac{a+b}{a-b},$$

$$\frac{u^2-uv+v^2+3\alpha^2}{3\alpha(u-v)}=\frac{a+b}{a-b},$$

$$\text{or } \frac{3(u-v)\ +(u+v)^2+12\alpha^2}{6(u-v)}=a+b, \quad \because\ 2\alpha=a-b.$$

$$\therefore\ 3(u-v)^2+(u+v)^2+3(a-b)^2=6(a+b)(u-v)$$

$$(u+v)^2+3\{(u-v)^2-2(a+b)(u-v)+(a+b)^2\}=12ab,$$

$$\therefore\ (u+v)^2+3(u-v-a-b)^2=12ab \ \ldots\ldots\ldots\ldots\ldots \ (3).$$

$$\text{From (2), } x^2-(y+a+b)^2=-2ab,$$

$$\therefore\ (x+y+a+b)(x-y-a-b)=-2ab,$$

$$\text{or } (u+v)(u-v-a-b)=-2ab \ \ldots\ldots\ldots \ (4).$$

From (3) and (4), multiplying (4) by $2\sqrt{3}$, adding and subtracting,

$$\left.\begin{aligned} u+v+\sqrt{3}(u-v-a-b)&=\pm 2\sqrt{ab}(3-\sqrt{3})^{\frac{1}{2}},\\ u+v-\sqrt{3}(u-v-a-b)&=\pm 2\sqrt{ab}(3+\sqrt{3})^{\frac{1}{2}},\end{aligned}\right\}$$

$$\text{or } \left.\begin{aligned} (1+\sqrt{3})x+(1-\sqrt{3})y&=(a+b)(\sqrt{3}-1)\pm 2\sqrt{ab}(3-\sqrt{3})^{\frac{1}{2}},\\ (1-\sqrt{3})x+(1+\sqrt{3})y&=-(a+b)(\sqrt{3}+1)\pm 2\sqrt{ab}(3+\sqrt{3})^{\frac{1}{2}},\end{aligned}\right\}$$

two *simple* equations for determining x and y.

5. Lady Sadler's lecturer, being in haste, had to ascend his staircase of three flights, of which the number of steps in the lower flight : that in the upper :: 8 : 9; and half that in the lower flight is greater by 2 than *one-seventh* of the whole staircase. He begins running at 3 times his usual pace of ascent, which would take him up the upper flight in $13\frac{1}{2}$ seconds; but for the middle and upper flights his pace becomes diminished in the ratio 6 : 5 : 4. When he had ascended *two thirds* of the upper flight, he had to wait 31 seconds to recover his breath; *walks* the

remainder at the rate of 1 step less than usual in 5 seconds; and discovers that he has been longer about it than if had *walked* up in the ratio of 69 : 50. What was the number of steps in the staircase?

Let x = N°. of steps he *usually* takes in $1''$,

$\therefore\ 13\frac{1}{2}.x$ = N°. of steps in the upper flight,

$\frac{8}{9}\times 13\frac{1}{2}.x = 12x$ = N°. of steps in lower flight,

$(6x-2)\times 7$ = N°. of steps in whole staircase,

$\therefore$ N°. in 2nd flight $= 7(6x-2) - 13\frac{1}{2}.x - 12x$,

$$= 16\tfrac{1}{2}.x - 14;$$

the usual time of ascent $= \dfrac{7(6x-2)}{x}$;

time actually taken to ascend 1st flight $= \dfrac{12x}{3x} = 4$,

.... 2nd .... $= \dfrac{16\frac{1}{2}.x.14}{2\frac{1}{2}.x} = \dfrac{33}{5} - \dfrac{28}{5x}$,

.... $\frac{2}{3}$ of 3rd .... $= \dfrac{\frac{2}{3}.13\frac{1}{2}.x}{2x} = \dfrac{9}{2}$,

.... remaining $\frac{1}{3}$ .... $= \dfrac{\frac{1}{3}.13\frac{1}{2}.x}{x-\frac{1}{5}} = \dfrac{\frac{9}{2}x}{x-\frac{1}{5}}$.

Then $4 + \dfrac{33}{5} - \dfrac{28}{5x} + \dfrac{9}{2} + \dfrac{\frac{9}{2}x}{x-\frac{1}{5}} + 31 = \dfrac{69}{50}.\dfrac{7(6x-2)}{x}$,

$$\frac{\frac{9}{2}x}{x-\frac{1}{5}} - \frac{28}{5x} + 46\tfrac{1}{10} = \frac{69}{50}.\frac{7(6x-2)}{x},$$

$$\frac{45x}{2(5x-1)} - \frac{28}{5x} + \frac{461}{10} = \frac{483}{50}.\frac{6x-2}{x},$$

$$\frac{45x}{2(5x-1)} = \frac{28}{5x} - \frac{461}{10} + \frac{483}{50}\left(6-\frac{2}{x}\right),$$

$$= -\frac{343}{25x} + \frac{593}{50},$$

$$\therefore \frac{45x}{5x-1} = \frac{593x-686}{25x},$$

$$\frac{45x}{5x-1} - 9 = \frac{593x-686}{25x} - 9,$$

$$\frac{9}{5x-1} = \frac{368x-686}{25x} = \frac{368(x-2)+50}{25x},$$

$$\frac{368(x-2)}{25x} = \frac{2}{x} - \frac{9}{5x-1} = \frac{x-2}{x(5x-1)},$$

$$\therefore x-2=0, \quad \text{or} \quad \frac{368}{25} = \frac{1}{5x-1},$$

$$\therefore x=2, \quad \text{or \&c.}$$

Hence N° of steps required $=7(6x-2)=70$.

6. A room is to be papered with paper 2 feet in width at 8*d*. a yard, and carpeted with carpet at 3*s*. 9*d*. per *square yard;* but *one-sixth* of the surface of the walls is taken up by a door and windows. Now the dimensions of the room are taken by an inaccurate foot-rule; hence the estimated cost is less than the actual cost in the proportion of 1 : 1·0201. Had however the Rule been 4 times as much *less* than a foot as it was *more* than a foot, the estimated cost in this case would have exceeded the actual cost by 17*s*. $0\frac{1}{2}$*d*. Having given that the cost of the carpet is double that of the paper, and that a square whose side is equal to the diagonal of the room contains 84 yards, find the dimensions of the room.

Let x, y, z, be the length, breadth, and height, of the room, in feet, respectively; and let the foot-rule be $1+n$ feet in length, (n being a fraction),

$$\therefore \frac{x}{1+n}, \frac{y}{1+n}, \frac{z}{1+n}, \text{ are the estimated dimensions;}$$

$$\therefore \text{actual cost} = (1+n)^2 \times \text{estimated cost},$$

$$\therefore (1+n)^2 = 1{\cdot}0201, \qquad \therefore n = {\cdot}01.$$

$$\text{The area of the 4 walls} = 2z(x+y),$$

$$\therefore \text{area to be papered} = \frac{5z}{3}(x+y),$$

$$\therefore \text{N° of yards of paper required} = \frac{5z}{18}(x+y),$$

$$\therefore \text{ cost of this in pence} = \frac{20z}{9}(x+y),$$

$$\text{and cost of carpeting } \ldots\ldots = \frac{xy}{9}\times 45 = 5xy,$$

$$\therefore \text{ total actual cost} = 5xy + \frac{20z}{9}(x+y);$$

$$\text{and, in supposed case, estimated cost} = \frac{1}{(1-4n)^2}\left\{5xy + \frac{20z}{9}(x+y)\right\},$$

$$\therefore \left\{\frac{1}{(1-4n)^2} - 1\right\}\cdot\left\{5xy + \frac{20z}{9}(x+y)\right\} = 204\tfrac{1}{6},$$

$$\therefore \left\{\left(\frac{25}{24}\right)^2 - 1\right\}\cdot\left\{5xy + \frac{20z}{9}(x+y)\right\} = \frac{1225}{6},$$

$$\therefore xy + \frac{4z}{9}(x+y) = \frac{245}{6}\times\frac{576}{49} = 480, \ldots\ldots (1).$$

Again, since carpeting costs double the papering,

$$5xy = \frac{40z}{9}(x+y) \ldots\ldots (2).$$

And, by question, $x^2+y^2+z^2=9\times 84=756 \;\ldots\; (3).$

From (1) and (2), $xy=320, \quad z(x+y)=360,$

$$\therefore 2(xy+xz+yz)=1360,$$

$$\therefore (x+y+z)^2=2116,$$

$$x+y+z=46,$$

$$\therefore 360+z^2=46z, \quad \text{from which } z=36, \text{ or } 10.$$

If $z=36$, x and y are impossible,

$$\therefore z=10, \qquad \left.\begin{array}{r} x+y=36, \\ xy=320, \end{array}\right\} \qquad \therefore \left.\begin{array}{r} x=20, \\ y=16, \end{array}\right\}$$

and dimensions required are 20, 16, 10, feet.

7. n men $A_1, A_2, A_3, \ldots A_n$, enter into partnership for nm years, the capital of the firm at the beginning of the $(m+1)^{\text{th}}$ year being £P; and they agree, after deducting their respective expenditures at the end of each year from their gain in that year to employ the remainder in the trade. Now the gain of the firm and the sum of the expenditures of $A_1, A_2, \ldots\ldots A_n$, in any year

are proportional to the capital employed in that year; and the separate expenditures of A_1, A_2,... in any year are proportional to their respective gains in that year. Supposing A_1's capital at beginning of the $(m+1)^{\text{th}}$ year : A_2's at beginning of the $(2m+1)^{\text{th}}$: A_3's at beginning of the $(3m+1)^{\text{th}}$... as $a_1 : a_2 : a_3 \ldots$; and that the ratio of the sum of the expenditures of A_1, $A_2 \ldots$ during the $(r-1)^{\text{th}}$ and $(r+1)^{\text{th}}$ periods of m years to that of their expenditure during the r^{th} period of m years is $a + \frac{1}{a} : 1$; find the original capitals of the several partners.

Let $x_1, x_2, \ldots x_n$, be the original capitals of $A_1, A_2, \ldots A_n$. Then, $\because$ gain of the *firm* in any year is proportional to the *whole* capital employed that year, the separate gains of A_1, A_2,... in any year will be proportional to their respective capitals in that year.

$\therefore$ $ux_1, ux_2, ux_3, \ldots$ will express the gains of $A_1, A_2, \ldots$ in 1st year,

vx_1, vx_2, vx_3, ... expenditures .... ,

$\therefore$ $(1+u-v)x_1$, $(1+u-v)x_2, \ldots$ are capitals of $A_1, A_2, \ldots$ for 2nd year.

The gains in 2nd year are $u(1+u-v)x_1$, $u(1+u-v)x_2$, &c.

... expend$^{\text{s}}$. .... $v(1+u-v)x_1$, $v(1+u-v)x_2$, &c.

capitals for 3rd year are $(1+u-v)^2x_1$, $(1+u-v)^2x_2$, &c.; and so on.

$\therefore$ capitals for $(m+1)^{\text{th}}$ year are $(1+u-v)^m x_1$, $(1+u-v)^m x_2$, &c.

Write V for $1+u-v$; then, by question,

$$\frac{V^m x_1}{a_1} = \frac{V^{2m} x_2}{a_2} = \frac{V^{3m} x_3}{a_3} = \text{\&c.} = \frac{V^{nm} x_n}{a_n} \ldots\ldots (1).$$

Also sum of expenditures of A_1, A_2,... during the $(r-1)^{\text{th}}$ period of m years is

$$v(x_1+x_2+\ldots+x_n)\{V^{m(r-2)} + V^{m(r-2)+1} + V^{m(r-2)+2} + \ldots + V^{m(r-1)-1}\}$$
$$= v(x_1+x_2+\ldots+x_n)V^{m(r-2)}\{1+V+V^2+\ldots+V^{m-1}\}.$$

Similarly, the expenditures during r^{th} and $\overline{r+1}^{\text{th}}$ periods are

$$v(x_1+x_2+\ldots+x_n)V^{m(r-1)}\{1+V+V^2+\ldots+V^{m-1}\},$$

and $v(x_1+x_2+\ldots+x_n)V^{mr}\{1+V+V^2+\ldots+V^{m-1}\}$.

$\therefore$ by question, $\dfrac{V^{m(r-2)} + V^{mr}}{V^{m(r-1)}} = a + \dfrac{1}{a}$,

$$\text{or}\quad \frac{V^{2m}+1}{V^m}=\frac{a^2+1}{a},$$

$$\frac{V^m+1}{V^m-1}=\frac{a+1}{a-1},\quad \therefore\ V=a^{\frac{1}{m}}.$$

$$\therefore\ \text{From (1)},\ \frac{ax_1}{a_1}=\frac{a^2x_2}{a_2}=\frac{a^3x_3}{a_3}=\&\text{c.}=\frac{a^nx_n}{a^n}=y,\ \text{suppose},$$

$$\therefore\ a\,(x_1+x_2+\ldots+x_n)=y\left(a_1+\frac{a_2}{a}=\frac{a_3}{a^2}+\ldots+\frac{a_n}{a^{n-1}}\right).$$

Now, by question, $V^m(x_1+x_2+\ldots+x_n)=P$,

$$\therefore\ y=\frac{Pa^{n-1}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n},$$

$$\therefore\ x_1=\frac{Pa_1a^{n-2}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n},$$

$$x_2=\frac{Pa_2a^{n-3}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n},$$

$$\&\text{c.}=\qquad\&\text{c.}$$

$$x_n=\frac{Pa_na^{-1}}{a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n}.$$

XII.

1. $x^2(m^2-n^2)+x\{m(2am-p)-n(2bn-p)\}$
$=(am-bn)(p-am-bn)$; find x.

$$m^2(x^2+2ax)-n^2(x^2-2bx)-px(m-n)=p(am-bn)-a^2m^2+b^2n^2,$$

$$m^2(x+a)^2-n^2(x+b)^2=p\{m(x+a)-n(x+b)\};$$

$$\therefore\ m(x+a)-n(x+b)=0,\ \text{or}\ m(x+a)+n(x+b)=p;$$

$$\therefore\ x=\frac{bn-am}{m-n},\ \text{or}\ \frac{p-am-bn}{m+n}.$$

2. $\dfrac{(x+a)(x+b)}{yz}=\dfrac{(y+a)(y+b)}{xz}=\dfrac{(z+a)(z+b)}{xy}=m$; find x, y, and z.

$$x(x+a)(x+b)=y(y+a)(y+b),$$

$$x^3-y^3+(a+b)(x^2-y^2)+ab(x-y)=0;$$

$$\left.\begin{array}{l}\therefore\ x=y, \text{ or } x^2+xy+y^2+(a+b)(x+y)+ab=0.\\ \text{Similarly } y=z, \text{ or } y^2+yz+z^2+(a+b)(y+z)+ab=0.\end{array}\right\}(\alpha).$$

1st. If $x=y=z$, then $(x+a)(x+b)=mx^2$, from which

$$x=\frac{a+b}{2(m-1)}\pm\sqrt{\frac{ab}{m-1}+\frac{(a+b)^2}{4(m-1)^2}};$$

$$\therefore\ x=y=z=\frac{1}{2(m-1)}\{a+b\pm\sqrt{a^2+b^2+2(2m-1)ab}\}.$$

2nd. If $x=y$, but not z,

$$\left.\begin{array}{ll}\text{then} & y^2+yz+z^2+(a+b)(y+z)+ab=0,\\ \text{then} & (z+a)(z+b)=my^2,\end{array}\right\}$$

from which $y^2+yz+(a+b)y+my^2=0$;

$\therefore\ y=0$, or $y+z+a+b+my=0$;

from $y=0$, we get $z=-a$, or $-b$;

from the other value of y, $-z=(m+1)y+a+b$.

Then $(z+a)(z+b)=my^2$ gives $my^2=(\overline{m+1}.y+b)(\overline{m+1}.y+a)$;

$$\therefore\ y^2+\frac{m+1}{m^2+m+1}(a+b)y=-\frac{1}{m^2+m+1}ab;$$

$$\therefore\ y=-\frac{1}{2(m^2+m+1)}\{(m+1)(a+b)\pm\sqrt{(m+1)^2(a-b)^2+4mab}\},$$

$$z=-(a+b)+\frac{m+1}{2(m^2+m+1)}\{(m+1)(a+b)\pm\sqrt{(m+1)^2(a-b)^2+4mab}\},$$

and $x=y$.

3rd. If x, y, z are unequal, then from (α), by subtraction and division,

$$\left.\begin{array}{c}x+z+y+a+b=0,\\ x^2+xy+y^2+(a+b)(x+y)+ab=0,\\ (z+a)(z+b)=mxy,\end{array}\right\}$$

from which we get, $(x+y+b)(x+y+a)=mxy$, from 1st and 3rd;

$$\therefore\ (x+y)^2+(a+b)(x+y)+ab=mxy,$$

and from 2nd, $\therefore\ xy=mxy$, and $\therefore\ x=0$, or $y=0$.

In either case $z=-a$, or $-b$; $\therefore$ the solutions are

$$\left.\begin{array}{l|c|c|c} x=\ 0 & 0 & -b & -a \\ y=-b & -a & 0 & 0 \\ z=-a & -b & -a & -b \end{array}\right\}.$$

3. $$\left.\begin{array}{l} x^2+xy+y^2=m(1-2\sqrt{3})x+5my\ldots\ldots(1) \\ x^2-xy=5mx+m(1-2\sqrt{3})y\ldots\ldots(2) \end{array}\right\}\text{ find } x \text{ and } y.$$

Multiply (2) by n, and add; then

$$(1+n)x^2+(1-n)xy+y^2=mx(1-2\sqrt{3}+5n)$$
$$+my\{5+n(1-2\sqrt{3})\}.$$

Now, if the 1st side be a perfect square,

$$(1-n)^2=4(1+n),\ \text{whence } n=3+2\sqrt{3},$$

and we have $(4+2\sqrt{3})x^2-(2+2\sqrt{3})xy+y^2$, i.e. $(\overline{\sqrt{3}+1}.x-y)^2$

$$=mx(16+8\sqrt{3})-my(4+4\sqrt{3}),$$
$$=4m\{(4+2\sqrt{3})x-(1+\sqrt{3})y\},$$
$$=4(\sqrt{3}+1)m\{(\sqrt{3}+1)x-y\};$$

$$\therefore (\sqrt{3}+1)x=y,\ \text{ or }\ (\sqrt{3}+1)x-y=4m(\sqrt{3}+1).$$

Taking the 1st, by (2),

$$x^2-(\sqrt{3}+1)x^2=5mx+m(1-2\sqrt{3})(\sqrt{3}+1)x,$$
$$\therefore\ x=0,\ \text{or } m;\ \text{and } y=0,\ \text{or } (\sqrt{3}+1)m.$$

Taking the 2nd, $y=(\sqrt{3}+1)(x-4m)$;

$\therefore$ from (2), $x^2-(\sqrt{3}+1)(x^2-4mx)$

$$=5mx+m(1-2\sqrt{3})(\sqrt{3}+1)(x-4m);$$

$$\therefore\ x=\frac{m}{2}\left(5+\frac{4}{\sqrt{3}}\right)\pm m\sqrt{\frac{1}{12}(43-40\sqrt{3})},$$

$$\text{and } y=\frac{m}{2}\left(1-\frac{5}{\sqrt{3}}\right)\pm m\sqrt{-\frac{1}{6}(34+37\sqrt{3})}.$$

4. $$\left.\begin{aligned} (\sqrt{5}-1)(x+yz)(y+zx)(z+xy)&=\sqrt{5}+1\ldots\ldots(1),\\ (\sqrt{5}+1)(1-x^2)(1-y^2)(1-z^2)&=\sqrt{5}-1\ldots\ldots(2),\\ x^2+y^2+z^2+2xyz&=\sqrt{5}+1\ldots\ldots(3), \end{aligned}\right\}\ \text{find } x,\ y, \text{ and } z.$$

Write u for xyz, then (1) becomes

$$\frac{1}{u}(u+x^2)(u+y^2)(u+z^2)=\frac{\sqrt{5}+1}{\sqrt{5}-1},$$

$$\text{or}\quad u^2+u(x^2+y^2+z^2)+x^2y^2+x^2z^2+y^2z^2+u=\frac{1}{2}(3+\sqrt{5});$$

and (2) becomes

$$1-(x^2+y^2+z^2)+x^2y^2+x^2z^2+y^2z^2-u^2=\frac{1}{2}(3-\sqrt{5});$$

$$\text{subtracting,}\quad 2u^2+u-1+(u+1)(x^2+y^2+z^2)=\sqrt{5};$$

$$\text{or}\quad (u+1)\{2u-1+x^2+y^2+z^2\}=\sqrt{5}.$$

From (3), $2u-1+x^2+y^2+z^2=\sqrt{5}$;

$$\therefore\ u+1=1,\ \text{i.e. } u=0.$$

Hence, either $x=0$, or $y=0$, or $z=0$.

1st. Suppose $z=0$, then from (1), $x^2y^2=\frac{1}{2}(3+\sqrt{5})$,
from (3), $x^2+y^2=\sqrt{5}+1$,

$$\therefore\ (x^2+y^2)^2=4x^2y^2,\ \text{i.e. } (x^2-y^2)^2=0;\ \text{and } \therefore\ x=\pm y,$$

$$\text{and } \therefore\ x^2=\frac{1}{2}(\sqrt{5}+1);$$

$$\therefore\ x=\pm\sqrt{\frac{1}{2}(\sqrt{5}+1)}=y,\ \text{and } z=0.$$

2nd. Suppose $y=0$, then $x=\pm z=\pm\sqrt{\frac{1}{2}(\sqrt{5}+1)}$.

3rd. Suppose $x=0$, then $y=\pm z=\pm\sqrt{\frac{1}{2}(\sqrt{5}+1)}$.

5. The Cambridge University Musical Society consists of half and full (or performing) members, who have respectively 1 and 2 tickets each to the concerts: a permanent list of persons

invited is kept, which may be considered as arising from the full members (known to be even in number) assigning 1 and 2 names alternately. On the occasion of a concert each member applies for an invitation for a stranger; the last 20 are rejected, and still from the smallness of the room, 9 had to stand for every 20 seated. When the new Town-Hall is built, (to hold 3 times as many people as the present one), if we assume the Society to receive an addition of 40 half and 20 full members, and that each member shall be allowed 1 more ticket, and shall obtain an invitation for 1 and 2 strangers respectively, and that the Committee moreover will increase the permanent list in the ratio of 10 : 9, and will invite 50 strangers, the Hall will be filled, and there will be present 30 more strangers than twice the number on the permanent list. 50 full members are to be supposed to perform at the first concert, and to sit amongst the audience at the second. How many will the Hall hold?

Let the N°. of full members at 1st be x,

and .... half .... y.

Then the N°. on the permanent list is $\frac{3}{2}x$;

and the N°. of strangers $= x + y - 20$.

Then, since each full member has 2 tickets, and 50 full members *perform*,

$$\text{the N°. of people present} = 2x + y + \frac{3}{2}x + (x + y - 20) - 50,$$

$$= \frac{9}{2}x + 2y - 70;$$

$$\therefore \text{ N°. the Hall will hold} = \frac{20}{29}\left(\frac{9}{2}x + 2y - 70\right).$$

Afterwards, there are $(x+20)$ full members, and $y+40$ half members;

$$\therefore \text{ permanent list from this cause is } \frac{3}{2}(x+20),$$

and therefore the actual permanent list will be

$$\frac{10}{9} \times \frac{3}{2}(x+20), \text{ or } \frac{5}{3}(x+20);$$

$$\text{N°. of strangers} = \{2(x+20) + (y+40)\} + 50;$$

$$= 2x + y + 130.$$

And N°. of *performers* is $(x+20)-50$, or $x-30$;

$$\therefore \text{N°. in the hall} = 3(x+20)+2(y+40)+\frac{5}{3}(x+20)$$
$$+(2x+y+130)-(x-30),$$
$$=\frac{17}{3}x+3y+\frac{1000}{3};$$

this fills the new hall;

$$\therefore\ \frac{17}{3}x+3y+\frac{1000}{3}=3\times\frac{20}{29}\left(\frac{9}{2}x+2y-70\right)\ldots\ldots\ldots(1).$$

Also, $2x+y+130=2\times\frac{5}{3}(x+20)+30$;

$$\therefore\ y=\frac{4}{3}x-\frac{100}{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$$\therefore\ \frac{60}{29}\left(\frac{9}{2}x+\frac{8x-200}{3}-70\right)=\frac{17}{3}x+(4x-100)+\frac{1000}{3}$$

$$\frac{10}{29}\{27x+(16x-820)\}=\frac{1}{3}(29x+700),$$

$$\frac{43x-820}{29}=\frac{29x+700}{30},\ \text{from which } x=100.$$

$$\text{And } y=\frac{1}{3}(4x-100)=100.$$

And the present hall holds $\frac{20}{29}\left(\frac{9}{2}x+2y-70\right)=400$.

6. The perimeters of 3 rectangles A, B, C, are as $2:3:5$, and their areas as $2:3:7$. Now a rectangle, equal in area to A, B, C, together, and *similar* to B, contains as many linear inches in its perimeter, as A and B together contain square inches; also the perimeter of a rectangle, equal in area to the excess of B and C together over A, and *similar* to A, is 8 inches less than the perimeter of a square equal in area to A, B, C, together. Find the sides of the rectangles A, B, C.

Let $2p$, $3p$, $5p$, be the perimeters of A, B, C, respectively,

$2a$, $3a$, $7a$, ... areas ....;

$\therefore$ areas of A, B, C, together $=12a=4\times$area of B.

Hence a rectangle equal to A, B, and C, together, and similar to B, will have its sides *double* of those of B; and $\therefore$ its perimeter $=6p$;

$$\therefore \text{by question,}\ 6p=5a\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, side of square, equal to A, B, C, together $=\sqrt{12a}=2\sqrt{3a}$;

$$\therefore \text{ its perimeter} = 8\sqrt{3a}.$$

Also, a rectangle equal to $B+C-A$,

$$=7a+3a-2a=8a=4\times \text{area of } A;$$

$\therefore$ since it is *similar* to A, its sides are double those of A; and $\therefore$ its perimeter $=4p$;

$$\therefore 4p+8=8\sqrt{3a}. \quad \text{Or, by (1)}, \quad 5a+12=12\sqrt{3a};$$

$$\therefore a-\frac{12}{5}\sqrt{3a}+\frac{36\times 3}{23}=\frac{108}{25}-\frac{60}{25}=\frac{48}{25};$$

$$\therefore a=12, \text{ or } \frac{12}{25}.$$

Hence, if x and y be the sides of A, we have

$$\left.\begin{aligned} x+y=p=\frac{5a}{6}&=10, \\ xy=2a\quad&=24, \end{aligned}\right\} \text{ using the first value of } a, \text{ since the other makes } x \text{ and } y \text{ impossible.}$$

$$\therefore \text{ we have } x=6, \text{ or } 4, \text{ and } y=4, \text{ or } 6.$$

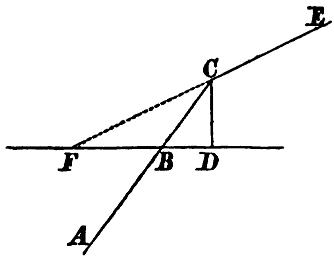
$\therefore$ sides of A are 6 inches, and 4 inches,

Similarly, ... B ... 12 ......, ... 3 ....

.... C ... 21 ......, ... 4 ....

7. A railway crosses obliquely a river flowing due East and West, and at a certain signal-post a branch turns off, making an angle with the main line equal to that made with the same by a road running due South from the junction to a wharf 1 mile from the bridge. A barge starts from the wharf at the same time as a train passes a station before it crosses the river; 45 min. afterwards the train is observed from the barge and found to be on the branch, 9 miles from the junction, and in a straight line with the signal-post, having passed the junction when the barge was under the bridge. On the return of the barge, which was towed at half its original rate (the stream being contrary), when it was $\frac{1}{\sqrt{6}}$ of a mile from the wharf, a horseman set out from the junction and arrived at the wharf simultaneously with the barge, having travelled at a rate which is the mean proportional between that of the train and the original rate of the barge. What was the rate of the train, and the distance of the station from the bridge?

Let A be the station, AC the railway, C the signal-post, CE the branch, FD the river, and D the wharf; E the point on the branch 9 miles from the junction, F being the intersection of EC produced with the line of the river, and CD perpendicular to FD, by the question.



Then, $\angle FCB = \angle BCD$, $BD = 1$ mile, $EC = 9$ miles.

Let x be the rate of the train in miles per hour,

y .... barge ....,

$\therefore \sqrt{xy} =$ .... horseman ....,

and let $AB = z$ miles, $BC = u$ miles; then

$$\frac{z+u}{x} = \text{time in } AC = \text{time in } DB = \frac{1}{y},$$

$$\text{or } z+u = \frac{x}{y} \text{.........(1)}.$$

$$\text{Also } \frac{BF}{y} = \text{time in } BF = \text{time in } CE = \frac{9}{x};$$

$$\therefore BF = 9\frac{y}{x}.$$

$$\text{And } BF : BD :: CF : CD; \quad \therefore BF = \frac{CF}{CD}, \quad \because BD = 1.$$

$$BF^2 = \frac{CF^2}{CD^2} = 1 + \frac{DF^2}{CD^2},$$

$$\therefore \left(9\frac{y}{x}\right)^2 = 1 + \frac{\left(1+9\frac{y}{x}\right)}{u^2-1} \text{.........(2)}.$$

$$\text{Also time in } DF = \frac{3}{4} \text{ hours};$$

$$\therefore 1+9\frac{y}{x} = \frac{3y}{4} \text{...............(3)}.$$

Again, time taken to *ride* $CD = \frac{\sqrt{u^2-1}}{\sqrt{xy}}$ hours, in which time the boat (on its return) goes $\frac{\sqrt{u^2-1}}{\sqrt{xy}} \cdot \frac{y}{2}$ miles;

$$\therefore \frac{1}{2}\sqrt{\frac{(u^2-1)y}{x}} = \frac{1}{\sqrt{6}}, \text{ or } \frac{(u^2-1)y}{x} = \frac{2}{3} \ldots\ldots (4).$$

From (2), $$0 = 1 - \left(9\frac{y}{x}\right)^2 + \frac{\left(1+9\frac{y}{x}\right)^2}{u^2-1},$$

$$\therefore 1 + 9\frac{y}{x} = 0, \text{ (which is inadmissible)},$$

$$\text{or } 0 = 1 - 9\frac{y}{x} + \frac{1+9\frac{y}{x}}{u^2-1};$$

$$\therefore \frac{9\frac{y}{x}+1}{9\frac{y}{x}-1} = u^2-1, \text{ or } 9\frac{y}{x} = \frac{u^2}{u^2-2};$$

$$\therefore \text{ from (4), } \frac{2}{3} = (u^2-1)\frac{1}{9} \cdot \frac{u^2}{u^2-2},$$

$$\therefore u^4 - u^2 = 6u^2 - 12;$$

$$\therefore u^2 = 4, \text{ or } 3.$$

Take the first value, then $u = 2$,

$$\text{and } \frac{y}{x} = \frac{1}{9} \cdot \frac{u^2}{u^2-2} = \frac{2}{9},$$

$$\text{and, from (3), } \frac{3y}{4} = 1 + 9\frac{y}{x} = 3;$$

$$\therefore y = 4; \text{ and } x = 18.$$

From (1), $z = -u + \frac{x}{y} = -2 + \frac{9}{2} = 2\frac{1}{2}$;

therefore rate of train = 18 miles per hour,

and distance of station A from the bridge = $2\frac{1}{2}$ miles.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

A CATALOGUE

OF

NEW WORKS IN GENERAL LITERATURE

PUBLISHED BY

LONGMAN, GREEN, LONGMAN, AND ROBERTS

39 PATERNOSTER ROW, LONDON.

CLASSIFIED INDEX

Agriculture and Rural Affairs.

Arts, Manufactures, and Architecture.

Biography.

Books of General Utility.

Botany and Gardening.

Chronology.

Commerce and Mercantile Affairs.

Criticism, History, and Memoirs.

Geography and Atlases.

Juvenile Books.

Medicine, Surgery, &c.

Miscellaneous Literature.

Natural History in general.

One-Volume Encyclopædias and Dictionaries.

Religious and Moral Works.

Poetry and the Drama.

The Sciences in general and Mathematics.

Rural Sports.

Veterinary Medicine, &c.

Voyages and Travels.

Works of Fiction.

ALPHABETICAL CATALOGUE

of

NEW WORKS and NEW EDITIONS

PUBLISHED BY

LONGMAN, GREEN, LONGMAN, AND ROBERTS,

PATERNOSTER ROW, LONDON.

Miss Acton's Modern Cookery for Private Families, reduced to a System of Easy Practice in a Series of carefully-tested Receipts, in which the Principles of Baron Liebig and other eminent writers have been as much as possible applied and explained. Newly-revised and enlarged Edition; with 8 Plates, comprising 27 Figures, and 150 Woodcuts. Fcp. 8vo. 7s. 6d.

Acton's English Bread-Book for Domestic Use, adapted to Families of every grade. Fcp. 8vo. price 4s. 6d.

The Afternoon of Life. By the Author of *Morning Clouds*. New and cheaper Edition, thoroughly revised. Fcp. 8vo. 5s.

Agassiz.—An Essay on Classification. By LOUIS AGASSIZ. 8vo. 12s.

Aikin's Select Works of the British Poets from Ben Jonson to Beattie. New Edition; with Biographical and Critical Prefaces, and Selections from recent Poets. 8vo. 18s.

Arago (F.)—Biographies of Distinguished Scientific Men. Translated by Admiral W. H. SMYTH, D.C.L., F.R.S., &c.; the Rev. BADEN POWELL, M.A.; and ROBERT GRANT, M.A., F.R.A.S. 8vo. 18s.

Arago's Meteorological Essays. With an Introduction by BARON HUMBOLDT. Translated under the superintendence of Lieut.-Col. E. SABINE, R.A., Treasurer and V.P.R.S. 8vo. 18s.

Arago's Popular Astronomy. Translated and edited by Admiral W. H. SMYTH, D.C.L., F.R.S.; and ROBERT GRANT, M.A., F.R.A.S. With 25 Plates and 358 Woodcuts. 2 vols. 8vo. price £2. 5s.

Arnold. — Merope, a Tragedy. By MATTHEW ARNOLD. With a Preface and an Historical Introduction. Fcp. 8vo. 5s.

Arnold.—Poems. By Matthew ARNOLD. FIRST SERIES, Third Edition. Fcp. 8vo. 5s. 6d. SECOND SERIES, price 5s.

Lord Bacon's Works. A New Edition, collected and edited by R. L. ELLIS, M.A., Fellow of Trinity College, Cambridge; J. SPEDDING, M.A. of Trinity College, Cambridge; and D. D. HEATH, Esq., Barrister-at-Law, and late Fellow of Trinity College, Cambridge. VOLS. I. to V. comprising the Division of *Philosophical Works*; with a copious INDEX. 5 vols. 8vo. price £4. 6s. VOL. VI. price 18s.

VOL VII., completing the Division of *Literary and Professional Works*, is just ready.

Joanna Baillie's Dramatic and Poetical Works: Comprising Plays of the Passions, Miscellaneous Dramas, Metrical Legends, Fugitive Pieces, and Ahalya Baee; with the Life of Joanna Baillie, Portrait and Vignette. Square crown 8vo. 21s. cloth; or 42s. morocco.

Baker.—The Rifle and the Hound in Ceylon. By S. W. BAKER, Esq. New Edition, with 13 Illustrations engraved on Wood. Fcp. 8vo. 4s. 6d.

Baker. — Eight Years' Wanderings in Ceylon. By S. W. BAKER, Esq. With 6 coloured Plates. 8vo. 15s.

Barth.—Travels and Discoveries in North and Central Africa: Being the Journal of an Expedition undertaken under the auspices of Her Britannic Majesty's Government in the Years 1849—1855. By HENRY BARTH, Ph.D., D.C.L., &c. With numerous Maps and Illustrations. 5 vols. 8vo. £5. 5s. cloth.

Bate. — Memoir of Captain W. Thornton Bate, R.N. By the Rev. JOHN BAILLIE, Author of "Memoirs of Hewitson," "Memoir of Adelaide Newton," &c. *New Edition;* with Portrait and 4 Illustrations. Fcp. 8vo. price 5s.

Bayldon's Art of Valuing Rents and Tillages, and Claims of Tenants upon Quitting Farms, at both Michaelmas and Lady-day; as revised by Mr. DONALDSON. *Seventh Edition,* enlarged and adapted to the Present Time. By ROBERT BAKER, Land-Agent and Valuer. 8vo. price 10s. 6d.

Bayldon's (R.) Treatise on Road Legislation and Management; with Remarks on Tolls, and on Repairing Turnpike-Roads and Highways. 8vo. price 3s. 6d.

Black's Practical Treatise on Brewing, based on Chemical and Economical Principles: With Formulæ for Public Brewers, and Instructions for Private Families. 8vo. 10s. 6d.

Blaine's Encyclopædia of Rural Sports; or, a complete Account, Historical, Practical, and Descriptive, of Hunting, Shooting, Fishing, Racing, &c. *New Edition,* revised and corrected to the Present Time; with above 600 Woodcut Illustrations, including 20 Subjects now added from Designs by JOHN LEECH. In One Volume, 8vo. price 42s. half-bound.

Bloomfield.—The Greek Testament: with copious English Notes, Critical, Philological, and Explanatory. Especially adapted to the use of Theological Students and Ministers. By the Rev. S. T. BLOOMFIELD, D.D., F.S.A. Ninth Edition, revised. 2 vols. 8vo. with Map, £2. 8s.

Dr. Bloomfield's College & School Edition of the Greek Testament: With brief English Notes, chiefly Philological and Explanatory. Seventh Edition; with Map and Index. Fcp. 8vo. 7s. 6d.

Dr. Bloomfield's College & School Lexicon to the Greek Testament. New Edition, revised. Fcp. 8vo. price 10s. 6d.

Boyd. — A Manual for Naval Cadets. Published with the sanction and approval of the Lords Commissioners of the Admiralty. By JOHN M'NEILL BOYD, Captain, R.N. With Compass-Signals in Colours, and 236 Woodcuts. Fcp. 8vo. 10s. 6d.

Bourne. — A Treatise on the Steam Engine, in its Application to Mines, Mills, Steam Navigation, and Railways. By the Artisan Club. Edited by JOHN BOURNE, C.E. New Edition; with 33 Steel Plates, and 349 Wood Engravings. 4to. 27s.

Bourne's Catechism of the Steam Engine in its various Applications to Mines, Mills, Steam Navigation, Railways, and Agriculture: With Practical Instructions for the Manufacture and Management of Engines of every class. Fourth Edition, enlarged; with 89 Woodcuts. Fcp. 8vo. 6s.

Brande's Dictionary of Science, Literature, and Art; comprising the History, Description, and Scientific Principles of every Branch of Human Knowledge; with the Derivation and Definition of all the Terms in general use. Third Edition, revised and corrected; with numerous Woodcuts. 8vo. 60s.

Professor Brande's Lectures on Organic Chemistry, as applied to Manufactures, including Dyeing, Bleaching, Calico Printing, Sugar Manufacture, the Preservation of Wood, Tanning, &c. Edited by J. SCOFFERN, M.B. Fcp. Woodcuts, 7s. 6d.

Brewer.—An Atlas of History and Geography, from the Commencement of the Christian Era to the Present Time: Comprising a Series of Sixteen Coloured Maps, arranged in Chronological Order, with Illustrative Memoirs. By the Rev. J. S. BREWER, M.A. *Second Edition,* revised and corrected. Royal 8vo. 12s. 6d. half-bound.

Brialmont. — The Life of the Duke of Wellington. From the French of ALEXIS BRIALMONT, Captain on the Staff of the Belgian Army: With Emendations and Additions. By the Rev. G. R. GLEIG, M.A., Chaplain-General to the Forces and Prebendary of St. Paul's. With Maps, Plans, and Portraits. VOLS. I. and II. 8vo. price 30s.

The THIRD and FOURTH VOLUMES (completion) are now in the press, and will take up the history of the Duke from the Battle of Waterloo, representing him as an Ambassador, as a Minister, and as a Citizen.

Brodie. — Psychological Inquiries, in a Series of Essays intended to illustrate the Influence of the Physical Organisation on the Mental Faculties. By Sir BENJAMIN C. BRODIE, Bart. Third Edition. Fcp. 8vo. 5s.

Dr. Bull on the Maternal Management of Children in Health and Disease. New Edition. Fcp. 8vo. 5s.

Dr. Bull's Hints to Mothers on the Management of their Health during the Period of Pregnancy and in the Lying-in Room: With an Exposure of Popular Errors in connexion with those subjects, &c.; and Hints upon Nursing. New Edition. Fcp. 8vo. 5s.

Dr. Bull's Work on Blindness, entitled the Sense of Vision Denied and Lost. Edited by the Rev. B. G. JOHNS, Chaplain of the Blind School, St. George's Fields. With a brief introductory Memoir of the Author by Mrs. BULL. Fcp. 8vo. 4s. 6d.

Bunsen.—Christianity and Mankind, their Beginnings and Prospects. By Baron C. C. J. BUNSEN, D.D., D.C.L., D.Ph. Being a New Edition, corrected, re-modelled, and extended, of *Hippolytus and his Age.* 7 vols. 8vo. £5. 5s.

*** This Edition is composed of three distinct works, as follows:—

1. Hippolytus and his Age; or, the Beginnings and Prospects of Christianity. 2 vols. 8vo. £1. 10s.
2. Outline of the Philosophy of Universal History applied to Language and Religion; containing an Account of the Alphabetical Conferences. 2 vols. 33s.
3. Analecta Ante-Nicœna. 3 vols. 8vo. £2. 2s.

Bunsen. — Lyra Germanica. Translated from the German by CATHERINE WINKWORTH. *Fifth Edition* of the FIRST SERIES, Hymns for the Sundays and Festivals of the Christian Year. *New Edition* of the SECOND SERIES, the Christian Life. Fcp. 8vo. 5s. each Series.

HYMNS from *Lyra Germanica*, 18mo. 1s.

*** These selections of German Hymns have been made from collections published in Germany by Baron BUNSEN; and form companion volumes to

Theologia Germanica: Which setteth forth many fair lineaments of Divine Truth, and saith very lofty and lovely things touching a Perfect Life. Translated by SUSANNA WINKWORTH. With a Preface by the Rev. CHARLES KINGSLEY; and a Letter by Baron BUNSEN. Third Edition. Fcp. 8vo. 5s.

Bunsen.—Egypt's Place in Universal History: An Historical Investigation, in Five Books. By Baron C. C. J. BUNSEN, D.C.L., D.Ph. Translated from the German by C. H. COTTRELL, Esq., M.A. With many Illustrations. VOL. I. 8vo. 28s.; VOL. II. price 30s.; and VOL. III. price 25s.

Bunting. — The Life of Jabez Bunting, D.D.: With Notices of contemporary Persons and Events. By his Son, THOMAS PERCIVAL BUNTING. VOL. I. with Two Portraits and a Vignette, in post 8vo. 7s. 6d.: or *(large paper and Proof Engravings)* in square crown 8vo. 10s. 6d.

Bunyan's Pilgrim's Progress: With a Preface by the Rev. CHARLES KINGSLEY, Rector of Eversley; and a Series of 126 Illustrations engraved on Steel and on Wood from Original Designs by CHARLES BENNETT. Fcp. 4to. price 21s. cloth, gilt edges.

Burton.—First Footsteps in East Africa; or, an Exploration of Harar. By RICHARD F. BURTON, Captain, Bombay Army. With Maps and coloured Plate. 8vo. 18s.

Burton.—Personal Narrative of a Pilgrimage to El Medinah and Meccah. By RICHARD F. BURTON, Captain, Bombay Army. *Second Edition*, revised; with coloured Plates and Woodcuts. 2 vols. crown 8vo. 24s.

Bishop Butler's Sketch of Modern and Ancient Geography. New Edition, thoroughly revised, with such Alterations introduced as continually progressive Discoveries and the latest information have rendered necessary. Post 8vo. 7s. 6d.

Bishop Butler's General Atlas of Modern and Ancient Geography; comprising Fifty-two full-coloured Maps; with complete Indices. New Edition, enlarged, and greatly improved. Edited by the Author's Son. Royal 4to. 24s.

The Cabinet Lawyer: A Popular Digest of the Laws of England, Civil and Criminal; with a Dictionary of Law Terms, Maxims, Statutes, and Judicial Antiquities; Correct Tables of Assessed Taxes, Stamp Duties, Excise Licenses, and Post-Horse Duties; Post-Office Regulations; and Prison Discipline. 18th Edition, comprising the Public Acts of the Session 1858. Fcp. 8vo. 10s. 6d.

The Cabinet Gazetteer: A Popular Exposition of All the Countries of the World. By the Author of *The Cabinet Lawyer.* Fcp. 8vo. 10s. 6d.

Caird. — Prairie Farming in America: With Notes by the way on Canada and the United States. By JAMES CAIRD, M.P., Author of "English Agriculture," "High Farming," &c. 16mo. 3s. 6d.

Calvert. — The Wife's Manual; or, Prayers, Thoughts, and Songs on Several Occasions of a Matron's Life. By the Rev. W. CALVERT, M.A. Ornamented from Designs by the Author in the style of *Queen Elizabeth's Prayer-Book.* Crown 8vo. 10s. 6d.

Catlow's Popular Conchology; or, the Shell Cabinet arranged according to the Modern System: With a detailed Account of the Animals, and a complete Descriptive List of the Families and Genera of Recent and Fossil Shells. Second Edition, improved; with 405 Woodcuts. Post 8vo. 14s.

Catz and Farlie's Book of Emblems.—Moral Emblems from JACOB CATZ and ROBERT FARLIE; with Aphorisms, Adages, and Proverbs of all Nations. The Illustrations freely rendered from designs found in the works of Catz and Farlie, by JOHN LEIGHTON, F.S.A., and engraved under his superintendence. Imperial 8vo. with 60 large Illustrations on Wood, and numerous Vignettes and Tail Pieces.

Cecil. — The Stud Farm; or, Hints on Breeding Horses for the Turf, the Chase, and the Road. Addressed to Breeders of Race-Horses and Hunters, Landed Proprietors, and Tenant Farmers. By CECIL. Fcp. 8vo. 5s.

Cecil's Stable Practice; or, Hints on Training for the Turf, the Chase, and the Road; with Observations on Racing and Hunting, Wasting, Race-Riding, and Handicapping: Addressed to all who are concerned in Racing, Steeple-Chasing, and Fox-Hunting. Second Edition. Fcp. 8vo. with Plate, 5s.

Chapman.—History of Gustavus Adolphus, and of the Thirty Years' War up to the King's Death: With some Account of its Conclusion by the Peace of Westphalia, in 1648. By B. CHAPMAN, M.A. 8vo. Plans, 12s. 6d.

Conington.—Handbook of Chemical Analysis, adapted to the Unitary System of Notation. By F. T. CONINGTON, M.A., F.C.S. Post 8vo. 7s. 6d. Also, *Tables of Qualitative Analysis,* designed as a Companion to the Handbook. Price 2s. 6d.

Connolly.—The Romance of the Ranks; or, Anecdotes, Episodes, and Social Incidents of Military Life. By T. W. J. CONNOLLY, Quartermaster of the Royal Engineers. 2 vols. 8vo. 21s.

Connolly's History of the Royal Sappers and Miners: Including the Services of the Corps in the Crimea and at the Siege of Sebastopol. *Second Edition;* with 17 coloured Plates. 2 vols. 8vo. 30s.

Conybeare and Howson's Life and Epistles of Saint Paul: Comprising a complete Biography of the Apostle, and a Translation of his Epistles inserted in Chronological Order. *Third Edition,* revised and corrected; with several Maps and Woodcuts, and 4 Plates. 2 vols. square crown 8vo. 31s. 6d.

*** The Original Edition, with more numerous Illustrations, in 2 vols. 4to. price 48s.—may also be had.

Dr. Copland's Dictionary of Practical Medicine: Comprising General Pathology, the Nature and Treatment of Diseases, Morbid Structures, and the Disorders especially incidental to Climates, to Sex, and to the different Epochs of Life; with numerous approved Formulæ of the Medicines recommended. Now complete in 3 vols. 8vo. price £5. 11s. cloth.

Bishop Cotton's Instructions in the Doctrine and Practice of Christianity. Intended as an Introduction to Confirmation. 4th Edition. 18mo. 2s. 6d.

Cresy's Encyclopædia of Civil Engineering, Historical, Theoretical, and Practical. Illustrated by upwards of 3,000 Woodcuts. *Second Edition,* revised; and extended in a Supplement, comprising Metropolitan Water-Supply, Drainage of Towns, Railways, Cubical Proportion, Brick and Iron Construction, Iron Screw Piles, Tubular Bridges, &c. 8vo. 63s.

Crosse. — Memorials, Scientific and Literary, of Andrew Crosse, the Electrician. Edited by Mrs. CROSSE. Post 8vo. 9s. 6d.

Crowe.—The History of France. By EYRE EVANS CROWE. In Five Volumes. VOL. I. 8vo. 14s.

Cruikshank. — The Life of Sir John Falstaff, illustrated in a Series of Twenty-four original Etchings by George Cruikshank. Accompanied by an imaginary Biography of the Knight, by ROBERT B. BROUGH. Royal 8vo. price 12s. 6d. cloth.

Lady Cust's Invalid's Own Book: A Collection of Recipes from various Books and various Countries. *Second Edition.* Fcp. 8vo. 2s. 6d.

The Rev. Canon Dale's Domestic Liturgy and Family Chaplain, in Two Parts: PART I. Church Services adapted for Domestic Use, with Prayers for Every Day of the Week, selected from the Book of Common Prayer; PART II. an appropriate Sermon for Every Sunday in the Year. Second Edition. Post 4to. 21s. cloth; 31s. 6d. calf; or £2. 10s. morocco.

Separately { THE FAMILY CHAPLAIN, 12s. THE DOMESTIC LITURGY, 10s. 6d.

Davy (Dr. J.)—The Angler and his Friend; or, Piscatory Colloquies and Fishing Excursions. By JOHN DAVY, M.D., F.R.S., &c. Fcp. 8vo. 6s.

By the same Author,

The Angler in the Lake District; or, Piscatory Colloquies and Fishing Excursions in Westmoreland and Cumberland. Fcp. 8vo. 6s. 6d.

De Fonblanque.—The Administration and Organisation of the British Army, with especial reference to Finance and Supply. By EDWARD BARRINGTON DE FONBLANQUE, Assistant Commissary-General. 8vo. 12s.

De la Rive's Treatise on Electricity in Theory and Practice. Translated for the Author by C. V. WALKER, F.R.S. 3 vols. 8vo. Woodcuts, £3. 13s.

Domenech.—Seven Years' Residence in the Great Deserts of North America. By the ABBE' DOMENECH. With a Map, and about Sixty Woodcut Illustrations. 2 vols. 8vo. [*Just ready.*

Abbe' Domenech's Missionary Adventures in Texas and Mexico: A Personal Narrative of Six Years' Sojourn in those Regions. 8vo. 10s. 6d.

The Eclipse of Faith; or, a Visit to a Religious Sceptic. *9th Edition.* Fcp. 8vo. 5s.

Defence of The Eclipse of Faith, by its Author: Being a Rejoinder to Professor Newman's *Reply*: Including a full Examination of that Writer's Criticism on the Character of Christ; and a Chapter on the Aspects and Pretensions of Modern Deism. *Second Edition*, revised. Post 8vo. 5s. 6d.

The Englishman's Greek Concordance of the New Testament: Being an Attempt at a Verbal Connexion between the Greek and the English Texts; including a Concordance to the Proper Names, with Indexes, Greek-English and English-Greek. New Edition, with a new Index. Royal 8vo. 42s.

The Englishman's Hebrew and Chaldee Concordance of the Old Testament: Being an Attempt at a Verbal Connexion between the Original and the English Translations; with Indexes, a List of the Proper Names and their Occurrences, &c. 2 vols. royal 8vo. £3. 13s. 6d.; large paper, £4. 14s. 6d.

Ephemera's Handbook of Angling; teaching Fly-fishing, Trolling, Bottom-Fishing, Salmon-Fishing: With the Natural History of River-Fish, and the best Modes of Catching them. Third Edition, corrected and improved; with Woodcuts. Fcp. 8vo. 5s.

Ephemera's Book of the Salmon: The Theory, Principles, and Practice of Fly-Fishing for Salmon; Lists of good Salmon Flies for every good River in the Empire; the Natural History of the Salmon, its Habits described, and the best way of artificially Breeding it. Fcp. 8vo. with coloured Plates, 14s.

Fairbairn.—Useful Information for Engineers: Being a Series of Lectures delivered to the Working Engineers of Yorkshire and Lancashire. By WILLIAM FAIRBAIRN, F.R.S., F.G.S. *Second Edition*; with Plates and Woodcuts. Crown 8vo. 10s. 6d.

First Impressions of the New World on Two Travellers from the Old in the Autumn of 1858. With Map by Arrowsmith. Post 8vo. 8s. 6d.

Fischer. — Francis Bacon of Verulam: Realistic Philosophy and its Age. By Dr. K. FISCHER. Translated by J. OXENFORD. Post 8vo. 9s. 6d.

Forester. — Rambles in the Islands of Corsica and Sardinia: With Notices of their History, Antiquities, and present Condition. By THOMAS FORESTER. With coloured Map; and numerous Lithographic and Woodcut Illustrations from Drawings made during the Tour by Lieut.-Col. M. A. Biddulph, R.A. Imperial 8vo. 28s.

Fraser. — Letters of Sir A. S. Fraser, K.C.B. Commanding the Royal Horse Artillery under the Duke of Wellington: Written during the Peninsular and Waterloo Campaigns. Edited by MAJOR-GENERAL SABINE, R.A. With Portrait, 2 Maps, and Plans. 8vo. 18s.

Freeman and Salvin.—Falconry: Its Claims, History, and Practice. By GAGE EARLE FREEMAN, M.A. ("Peregrine" of the *Field* newspaper); and Captain F. H. SALVIN. Post 8vo. with Woodcut Illustrations from Drawings by Wolf, price 10s. 6d. cloth.

Garratt.—Marvels and Mysteries of Instinct; or, Curiosities of Animal Life. By GEORGE GARRATT. *Second Edition*, improved. Fcp. 8vo. 4s. 6d.

Gilbart. — A Practical Treatise on Banking. By JAMES WILLIAM GILBART, F.R.S. *Sixth Edition*. 2 vols. 12mo. 16s.

Gilbart's Logic of Banking: A Familiar Exposition of the Principles of Reasoning, and their Application to the Art and the Science of Banking. 12mo. with Portrait, 12s. 6d.

Gleig. — Essays, Biographical, Historical, and Miscellaneous, contributed chiefly to the *Edinburgh* and *Quarterly Reviews*. By the Rev. G. R. GLEIG, M.A., Chaplain-General to the Forces, and Prebendary of St. Paul's. 2 vols. 8vo. price 21s.

The Poetical Works of Oliver Goldsmith. Edited by BOLTON CORNEY, Esq. Illustrated by Wood Engravings, from Designs by Members of the Etching Club. Square crown 8vo. cloth, 21s.; morocco, £1. 16s.

Gosse.—A Naturalist's Sojourn in Jamaica. By P. H. GOSSE, Esq. With Plates. Post 8vo. 14s.

Greathed.—Letters from Delhi written during the Siege. By H. H. GREATHED, late of the Bengal Civil Service. Edited by his Widow. Post 8vo. 8s. 6d.

Green.—Lives of the Princesses of England. By Mrs. MARY ANNE EVERETT GREEN, Editor of the *Letters of Royal and Illustrious Ladies*. With numerous Portraits. Complete in 6 vols. post 8vo. 10s. 6d. each.

Greyson.—Selections from the Correspondence of R. E. GREYSON, Esq. Edited by the Author of *The Eclipse of Faith*. New Edition. Crown 8vo. 7s. 6d.

Grove.—The Correlation of Phy-sical Forces. By W. R. GROVE, Q.C., M.A. *Third Edition*. 8vo. 7s.

Gurney.—St. Louis and Henri IV.: Being a Second Series of Historical Sketches. By the Rev. JOHN H. GURNEY, M.A. Fcp. 8vo. 6s.

Evening Recreations; or, Samples from the Lecture-Room. Edited by Rev. J. H. GURNEY. Crown 8vo. 5s.

Gwilt's Encyclopædia of Archi-tecture, Historical, Theoretical, and Practical. By JOSEPH GWILT. With more than 1,000 Wood Engravings, from Designs by J. S. GWILT. 8vo. 42s.

Hare (Archdeacon).—The Life of Luther, in Forty-eight Historical Engravings. By GUSTAV KÖNIG. With Explanations by Archdeacon HARE and SUSANNAH WINKWORTH. Fcp. 4to. 28s.

Harford.—Life of Michael Angelo Buonarroti: With Translations of many of his Poems and Letters; also Memoirs of Savonarola, Raphael, and Vittoria Colonna. By JOHN S. HARFORD, Esq., D.C.L., F.R.S. Second Edition, revised; with 20 Plates. 2 vols. 8vo. 25s.

Illustrations, Architectural and Pictorical, of the Genius of Michael Angelo Buonarroti. With Descriptions of the Plates, by the Commendatore CANINA; C. R. COCKERELL, Esq., R.A.; and J. S. HARFORD, Esq., D.C.L., F.R.S. Folio, 73s. 6d. half-bound.

Harrison. — The Light of the Forge; or, Counsels from the Sick-Bed of E.M. By the Rev. W. HARRISON, M.A., Domestic Chaplain to the Duchess of Cambridge. Fcp. 8vo. 5s.

Harry Hieover's Stable Talk and Table Talk; or, Spectacles for Young Sportsmen. New Edition, 2 vols. 8vo. Portrait, 24s.

Harry Hieover.—The Hunting-Field. By HARRY HIEOVER. With Two Plates. Fcp. 8vo. 5s. half-bound.

Harry Hieover. — Practical Horsemanship. *Second Edition;* with 2 Plates. Fcp. 8vo. 5s. half-bound.

Harry Hieover.—The Pocket and the Stud; or, Practical Hints on the Management of the Stable. By HARRY HIEOVER. Fcp. 8vo. Portrait, 5s.

Harry Hieover.—The Stud, for Practical Purposes and Practical Men: Being a Guide to the Choice of a Horse for use more than for show. Fcp. 5s.

Hassall.—A History of the British Freshwater Algæ: Including Descriptions of the Desmideæ and Diatomaceæ. By ARTHUR HILL HASSALL, M.D. 2 vols. 8vo. with 103 Plates, £1. 15s.

Hassall.—Adulterations Detected; or, Plain Instructions for the Discovery of Frauds in Food and Medicine. By ARTHUR HILL HASSALL, M.D. Lond., Analyst of *The Lancet* Sanitary Commission, and Author of the Reports of that Commission published under the title of *Food and its Adulterations* (which may also be had, in 8vo. price 28s.) With 225 Illustrations, engraved on Wood. Crown 8vo. 17s. 6d.

Col. Hawker's Instructions to Young Sportsmen in all that relates to Guns and Shooting. 11th Edition, revised by the Author's Son, Major P. W. L. HAWKER. With Portrait, Plates, and Woodcuts. Sq. crown 8vo. 18s.

Haydn's Book of Dignities: Containing Rolls of the Official Personages of the British Empire, Civil, Ecclesiastical, Judicial, Military, Naval, and Municipal, from the Earliest Periods to the Present Time. Together with the Sovereigns of Europe, from the Foundation of their respective States; the Peerage and Nobility of Great Britain, &c. 8vo. 25s.

Hayward. — Biographical and Critical Essays, reprinted from Reviews, with Additions and Corrections. By A. HAYWARD, Esq., Q.C. 2 vols. 8vo. 24s.

Sir John Herschel's Outlines of Astronomy. Fifth Edition, revised and corrected to the existing state of astronomical knowledge; with Plates and Woodcuts. 8vo. 18s.

Sir John Herschel's Essays from the *Edinburgh* and *Quarterly Reviews,* with Addresses and other Pieces. 8vo. 18s.

Hinchliff. — Summer Months among the Alps: With the Ascent of Monte Rosa. By THOS. W. HINCHLIFF, Barrister-at-Law. Post 8vo. 10s. 6d.

Hints on Etiquette and the Usages of Society: With a Glance at Bad Habits. New Edition, revised (with Additions) by a Lady of Rank. Fcp. 8vo. 2s. 6d.

Holland.—Medical Notes and Reflections. By Sir HENRY HOLLAND, M.D., F.R.S., &c., Physician in Ordinary to the Queen and Prince-Consort. Third Edition. 8vo. 18s.

Sir H. Holland's Chapters on Mental Physiology, founded chiefly on Chapters contained in *Medical Notes and Reflections.* Post 8vo. 8s. 6d.

Hooker.—Kew Gardens; or, a Popular Guide to the Royal Botanic Gardens of Kew. By Sir WILLIAM JACKSON HOOKER, K.H., &c., Director. With many Woodcuts. 16mo. 6d.

Hooker and Arnott's British Flora; comprising the Phænogamous or Flowering Plants, and the Ferns. Seventh Edition, with Additions and Corrections; and numerous Figures illustrative of the Umbelliferous Plants, the Composite Plants, the Grasses, and the Ferns. 12mo. with 12 Plates, 14s.; with the Plates coloured, 21s.

Horne's Introduction to the Critical Study and Knowledge of the Holy Scriptures. *Tenth Edition,* revised, corrected, and brought down to the present time. Edited by the Rev. T. HARTWELL HORNE, B.D. (the Author); the Rev. SAMUEL DAVIDSON, D.D. of the University of Halle, and LL.D.; and S. PRIDEAUX TREGELLES, LL.D. With 4 Maps and 22 Vignettes and Facsimiles. 4 vols. 8vo. £3. 13s. 6d.

Horne.—A Compendious Introduction to the Study of the Bible. By the Rev. T. HARTWELL HORNE, B.D. New Edition, with Maps, &c. 12mo. 9s.

Hoskyns.—Talpa; or, the Chronicles of a Clay Farm: An Agricultural Fragment. By CHANDOS WREN HOSKYNS, Esq. Fourth Edition. With 24 Woodcuts from Designs by GEORGE CRUIKSHANK. 16mo. 5s. 6d.

Howitt (A. M.)—An Art-Student in Munich. By ANNA MARY HOWITT. 2 vols. post 8vo. 14s.

Howitt.—The Children's Year. By MARY HOWITT. With Four Illustrations. Square 16mo. 5s.

Howitt. — Tallangetta, the Squatter's Home: A Story of Australian Life. By WILLIAM HOWITT. 2 vols. post 8vo. 18s.

Howitt. — Land, Labour, and Gold; or, Two Years in Victoria: With Visit to Sydney and Van Diemen's Land. By WILLIAM HOWITT. Second Edition. 2 vols. crown 8vo. 10s.

W. Howitt's Visits to Remarkable Places: Old Halls, Battle-Fields, and Scenes illustrative of Striking Passages in English History and Poetry. With about 80 Wood Engravings. *New Edition.* 2 vols. square crown 8vo. 25s.

William Howitt's Boy's Country Book: Being the Real Life of a Country Boy, written by himself; exhibiting all the Amusements, Pleasures, and Pursuits of Children in the Country. With 40 Woodcuts. Fcp. 8vo. 6s.

William Howitt's Rural Life of England. With Woodcuts by Bewick and Williams. Medium 8vo. 21s.

The Abbe' Huc's Work on the Chinese Empire, founded on Fourteen Years' Travel and Residence in China. People's Edition, with 2 Woodcut Illustrations. Crown 8vo. 5s.

Huc. — Christianity in China, Tartary, and Thibet. By M. l'Abbé HUC, formerly Missionary Apostolic in China. VOLS. I. and II. 8vo. 21s.; and VOL. III 10s. 6d.

Hudson's Executor's Guide. New and improved Edition; with the Statutes enacted, and the Judicial Decisions pronounced since the last Edition incorporated. Fcp. 8vo. 6s.

Hudson's Plain Directions for Making Wills in conformity with the Law. New Edition, corrected and revised by the Author; and practically illustrated by Specimens of Wills containing many varieties of Bequests, also Notes of Cases judicially decided since the Wills Act came into operation. Fcp. 8vo. 2s. 6d.

Hudson and Kennedy's Ascent of Mont Blanc by a New Route and Without Guides. *Second Edition*, with Plate and Map. Post 8vo. 5s. 6d.

Humboldt's Cosmos. Translated, with the Author's authority, by Mrs. SABINE. VOLS. I. and II. 16mo. Half-a-Crown each, sewed; 3s. 6d. each, cloth; or in post 8vo. 12s. each, cloth. VOL. III. post 8vo. 12s. 6d. cloth: or in 16mo. Part I. 2s. 6d. sewed, 3s. 6d. cloth; and Part II. 3s. sewed, 4s. cloth. VOL. IV. PART I. post 8vo. 15s. cloth; 16mo. 7s. 6d. cloth.

Humboldt's Aspects of Nature. Translated, with the Author's authority, by Mrs. SABINE. 16mo. price 6s.: or in 2 vols. 3s. 6d. each, cloth; 2s. 6d. each, sewed.

Humphreys.— Parables of Our Lord, illuminated and ornamented in the style of the Missals of the Renaissance by H. N. HUMPHREYS. Square fcp. 8vo. 21s. in massive carved covers; or 30s. bound in morocco, by Hayday.

Hunt (Capt.).—The Horse and his Master: With Hints on Breeding, Breaking, Stable-Management, Training, Elementary Horsemanship, Riding to Hounds, &c. By VERE D. HUNT, Esq., late 109th Regt. Co. Dublin Militia. Fcp. 8vo. with Frontispiece, price 5s.

Hunt.—Researches on Light in its Chemical Relations; embracing a Consideration of all the Photographic Processes. By ROBERT HUNT, F.R.S. Second Edition, with Plate and Woodcuts. 8vo. 10s. 6d.

Hutchinson. — Impressions of Western Africa: With a Report on the Peculiarities of Trade up the Rivers in the Bight of Biafra. By J. T. HUTCHINSON, Esq., British Consul for the Bight of Biafra and the Island of Fernando Po. Post 8vo. 8s. 6d.

Idle.—Hints on Shooting, Fishing, &c., both on Sea and Land, and in the Fresh-Water Lochs of Scotland: Being the Experiences of C. IDLE, Esq. Fcp. 8vo. 5s.

Mrs. Jameson's Two Lectures on the Social Employments of Women, *Sisters of Charity* and the *Communion of Labour*. New Edition, with a Prefatory Letter on the Present Condition and Requirements of the Women of England. Fcp. 8vo. 2s.

Mrs. Jameson's Legends of the Saints and Martyrs, as represented in Christian Art: Forming the First Series of *Sacred and Legendary Art*. Third Edition; with 17 Etchings and upwards of 180 Woodcuts. 2 vols. square crown 8vo. 31s. 6d.

Mrs. Jameson's Legends of the Monastic Orders, as represented in Christian Art. Forming the Second Series of *Sacred and Legendary Art*. Second Edition, enlarged; with 11 Etchings by the Author and 88 Woodcuts. Square crown 8vo. 28s.

Mrs. Jameson's Legends of the Madonna, as represented in Christian Art: Forming the Third Series of *Sacred and Legendary Art*. Second Edition, corrected and enlarged; with 27 Etchings and 165 Wood Engravings. Square crown 8vo. 28s.

Mrs. Jameson's Commonplace-Book of Thoughts, Memories, and Fancies, Original and Selected. *Second Edition*, revised and corrected; with Etchings and Woodcuts. Crown 8vo. price 18s.

Jaquemet's Compendium of Chronology: Containing the most important Dates of General History, Political, Ecclesiastical, and Literary, from the Creation of the World to the end of the Year 1854. Post 8vo. 7s. 6d.

Jaquemet's Chronology for Schools: Containing the most important Dates of General History, Political, Ecclesiastical, and Literary, from the Creation of the World to the end of the Year 1857. Fcp. 8vo. 3s. 6d.

Lord Jeffrey's Contributions to The Edinburgh Review. A New Edition, complete in One Volume, with Portrait and Vignette. Square crown 8vo. 21s. cloth; or 30s. calf.—Or in 3 vols. 8vo. price 42s.

Bishop Jeremy Taylor's Entire Works: With Life by Bishop Heber. Revised and corrected by the Rev. Charles Page Eden, Fellow of Oriel College, Oxford. Now complete in 10 vols. 8vo. 10s. 6d. each.

Kane.—Wanderings of an Artist among the Indians of North America; from Canada to Vancouver's Island and Oregon, through the Hudson's Bay Company's Territory, and back again. By Paul Kane. With Map, Illustrations in Colours, and Wood Engravings. 8vo. 21s.

Kemble.—The Saxons in England: A History of the English Commonwealth till the Conquest. By J. M. Kemble, M.A. 2 vols. 8vo. 28s.

Keith Johnston's Dictionary of Geography, Descriptive, Physical, Statistical, and Historical: Forming a complete General Gazetteer of the World. *Third Edition*, rectified to May 1859. In 1 vol. of 1,360 pages, comprising about 50,000 Names of Places, 8vo. 30s. cloth; or half-bound in russia, 35s.

Kesteven.—A Manual of the Domestic Practice of Medicine. By W. B. Kesteven, F.R.C.S.E., &c. Square post 8vo. 7s. 6d.

Kirby and Spence's Introduction to Entomology; or, Elements of the Natural History of Insects: Comprising an Account of Noxious and Useful Insects, of their Metamorphoses, Food, Stratagems, Habitations, Societies, Motions, Noises, Hybernation, Instinct, &c. *Seventh Edition*, with an Appendix relative to the Origin and Progress of the work. Crown 8vo. 5s.

A Lady's Tour round Monte Rosa; with Visits to the Italian Valleys of Anzasca, Mastalone, Camasco, Sesia, Lys, Challant, Aosta, and Cogne: In a Series of Excursions in the Years 1850, 1856, 1858. With Map, 4 Illustrations in Colours from Sketches by Mr. G. Barnard, and 8 Wood Engravings. Post 8vo. 14s.

Lardner's Cabinet Cyclopædia of History, Biography, Literature, the Arts and Sciences, Natural History, and Manufactures. A Series of Original Works by Eminent Writers. Complete in 132 vols. fcp. 8vo. with Vignette Titles, price £19. 19s. cloth lettered.

The Works *separately*, in single Volumes or Sets, price 3s. 6d. each Volume, cloth lettered.

Mrs. R. Lee's Elements of Na-tural History; or, First Principles of Zoology: Comprising the Principles of Classification, interspersed with amusing and instructive Accounts of the most remarkable Animals. New Edition; Woodcuts. Fcp. 8vo. 7s. 6d.

The Letters of a Betrothed. Fcp. 8vo. price 5s. cloth.

Letters to my Unknown Friends. By a Lady, Author of *Letters on Happiness*. *Fourth Edition*. Fcp. 8vo. 5s.

L.E.L. — The Poetical Works of Letitia Elizabeth Landon; comprising the *Improvisatrice*, the *Venetian Bracelet*, the *Golden Violet*, the *Troubadour*, and Poetical Remains. 2 vols. 16mo. 10s. cloth; morocco, 21s.

Dr. John Lindley's Theory and Practice of Horticulture; or, an Attempt to explain the principal Operations of Gardening upon Physiological Grounds: Being the Second Edition of the *Theory of Horticulture*, much enlarged; with 98 Woodcuts. 8vo. 21s.

Dr. John Lindley's Introduction to Botany. New Edition, with corrections and copious Additions. 2 vols. 8vo. with Plates and Woodcuts, 24s.

Dr. John Lindley's Synopsis of the British Flora arranged according to the Natural Orders; containing Vasculares or Flowering Plants. *Third Edition* (reprinted). Fcp. 8vo. 6s.

Linwood. — Anthologia Oxoniensis, sive Florilegium e Lusibus poeticis diversorum Oxoniensium Græcis et Latinis decerptum. Curante GULIELMO LINWOOD, M.A. 8vo. 14s.

Lorimer's Letters to a Young Master Mariner on some Subjects connected with his Calling. Fcp. 8vo. price 5s. 6d.

Loudon's Encyclopædia of Gardening: Comprising the Theory and Practice of Horticulture, Floriculture, Aboriculture, and Landscape-Gardening. With 1,000 Woodcuts. 8vo. 31s. 6d.

Loudon's Encyclopædia of Trees and Shrubs, or *Arboretum et Fructicetum Britannicum* abridged: Containing the Hardy Trees and Shrubs of Great Britain, Native and Foreign, Scientifically and Popularly Described. With about 2,000 Woodcuts. 8vo. 50s.

Loudon's Encyclopædia of Agriculture: Comprising the Theory and Practice of the Valuation, Transfer, Laying-out, Improvement, and Management of Landed Property, and of the Cultivation and Economy of the Animal and Vegetable Productions of Agriculture. With 1,100 Woodcuts. 8vo. 31s. 6d.

Loudon's Encyclopædia of Plants: Comprising the Specific Character, Description, Culture, History, Application in the Arts, and every other desirable Particular respecting all the Plants found in Great Britain. With upwards of 12,000 Woodcuts. 8vo. price £3. 13s. 6d.

Loudon's Encyclopædia of Cottage, Farm, and Villa Architecture and Furniture. New Edition, edited by Mrs. LOUDON; with more than 2,000 Woodcuts. 8vo. 63s.

Loudon's Hortus Britannicus; or, Catalogue of all the Plants found in Great Britain. New Edition, corrected by Mrs. LOUDON. 8vo. 31s. 6d.

Mrs. Loudon's Lady's Country Companion; or, How to Enjoy a Country Life Rationally. Fourth Edition. Fcp. 8vo. 5s.

Mrs. Loudon's Amateur Gardener's Calendar, or Monthly Guide to what should be avoided and done in a Garden. Second Edition, revised. Crown 8vo. with Woodcuts, 7s. 6d.

Low's Elements of Practical Agriculture; comprehending the Cultivation of Plants, the Husbandry of the Domestic Animals, and the Economy of the Farm. New Edition; with 200 Woodcuts. 8vo. 21s.

Macaulay. — Speeches of the Right Hon. Lord MACAULAY. Corrected by HIMSELF. 8vo. 12s.

Macaulay. — The History of England from the Accession of James II. By the Right Hon. Lord MACAULAY. New Edition. Vols. I. and II. 8vo. 32s.; Vols. III. and IV. 36s.

Lord Macaulay's History of England from the Accession of James II. New Edition of the first Four Volumes of the Octavo Edition, revised and corrected. 7 vols. post 8vo. 6s. each.

Lord Macaulay's Critical and Historical Essays contributed to The Edinburgh Review. Four Editions:—

1. A LIBRARY EDITION (the *Eighth*), in 3 vols. 8vo. price 36s.
2. Complete in ONE VOLUME, with Portrait and Vignette. Square crown 8vo. price 21s. cloth; or 30s. calf.
3. Another NEW EDITION, in 3 vols. fcp. 8vo. price 21s. cloth.
4. The PEOPLE'S EDITION, in 2 vols. crown 8vo. price 8s. cloth.

Macaulay. — Lays of Ancient Rome, with *Ivry* and the *Armada*. By the Right Hon. Lord MACAULAY. New Edition. 16mo. price 4s. 6d. cloth; or 10s. 6d. bound in morocco.

Lord Macaulay's Lays of Ancient Rome. With numerous Illustrations, Original and from the Antique, drawn on Wood by George Scharf, jun. Fcp. 4to. 21s. boards; or 42s. bound in morocco.

Mac Donald.—Poems. By George MAC DONALD, Author of *Within and Without*. Fcp. 8vo. 7s.

Mac Donald. — Within and Without: A Dramatic Poem. By GEORGE MAC DONALD. Fcp. 8vo. 4s. 6d.

Mac Dougall. — The Theory of War illustrated by numerous Examples from History. By Lieutenant-Colonel MAC DOUGALL, Commandant of the Staff College. *Second Edition*, revised. Post 8vo. with Plans, 10s. 6d.

Mac Dougall. — The Campaigns of Hannibal, arranged and critically considered, expressly for the use of Students of Military History. By Lieut.-Col. P. L. MAC DOUGALL, Commandant of the Staff College. Post 8vo. 7s. 6d.

M'Dougall. — The Eventful Voyage of *H.M. Discovery Ship* Resolute *to the Arctic Regions in search of Sir John Franklin and the Missing Crews of H.M. Discovery Ships* Erebus *and* Terror, 1852, 1853, 1854. By GEORGE F. M'DOUGALL, Master. With a coloured Chart, Illustrations in Lithography, and Woodcuts. 8vo. 21s.

Sir James Mackintosh's Miscellaneous Works: Including his Contributions to The Edinburgh Review. Complete in One Volume; with Portrait and Vignette. Square crown 8vo. 21s. cloth; or 30s. bound in calf: or in 3 vols. fcp. 8vo. 21s.

Sir James Mackintosh's History of England from the Earliest Times to the final Establishment of the Reformation. 2 vols. 8vo. 21s.

M'Culloch's Dictionary, Practical, Theoretical, and Historical, of Commerce, and Commercial Navigation. Illustrated with Maps and Plans. New Edition, revised and adapted to the Present Time.

[*Just ready*.

M'Culloch's Dictionary, Geographical, Statistical, and Historical, of the various Countries, Places, and principal Natural Objects in the World. Illustrated with Six large Maps. New Edition, revised. 2 vols. 8vo. 63s.

Maguire. — Rome; its Ruler and its Institutions. By JOHN FRANCIS MAGUIRE, M.P. *Second Edition*, enlarged; with a new Portrait of Pope Pius IX. Post 8vo. 10s. 6d.

Mrs. Marcet's Conversations on Natural Philosophy, in which the Elements of that Science are familiarly explained. Thirteenth Edition, enlarged and corrected; with 34 Plates. Fcp. 8vo. price 10s. 6d.

Mrs. Marcet's Conversations on Chemistry, in which the Elements of that Science are familiarly explained and illustrated by Experiments. New Edition, improved. 2 vols. fcp. 8vo. 14s.

Marshman.—The Life and Times of Carey, Marshman, and Ward: Embracing the History of the Serampore Mission. By JOHN CLARK MARSHMAN. 2 vols. 8vo. 25s.

Martineau. — Studies of Christianity: A Series of Original Papers, now first collected, or New. By JAMES MARTINEAU. Crown 8vo. 7s. 6d.

Martineau. — Endeavours after the Christian Life: Discourses. By JAMES MARTINEAU. 2 vols. post 8vo. price 7s. 6d. each.

Martineau. — Hymns for the Christian Church and Home. Collected and edited by JAMES MARTINEAU. *Eleventh Edition*, 12mo. 3s. 6d. cloth, or 5s. calf; *Fifth Edition*, 32mo. 1s. 4d. cloth, or 1s. 8d. roan.

Martineau.—Miscellanies: Comprising Essays chiefly religious and controversial. By JAMES MARTINEAU. Crown 8vo. 9s.

Maunder's Scientific and Literary Treasury: A new and popular Encyclopædia of Science and the Belles-Lettres; including all Branches of Science, and every subject connected with Literature and Art. Fcp. 8vo. 10s.

Maunder's Biographical Treasury; consisting of Memoirs, Sketches, and brief Notices of above 12,000 Eminent Persons of All Ages and Nations, from the Earliest Period of History: Forming a complete Dictionary of Universal Biography. *Eleventh Edition*, corrected and extended in a Supplement to the Present Time. Fcp. 8vo. 10s.

Maunder's Treasury of Knowledge, and Library of Reference; comprising an English Dictionary and Grammar, a Universal Gazetteer, a Classical Dictionary, a Chronology, a Law Dictionary, a Synopsis of the Peerage, numerous useful Tables, &c. New Edition, reconstructed by B. B. WOODWARD, B.A.; assisted by J. MORRIS, Solicitor, and W. HUGHES, F.R.G.S. Fcp. 8vo. 10s.

Maunder's Treasury of Natural History; or, a Popular Dictionary of Animated Nature: In which the Zoological Characteristics that distinguish the different Classes, Genera, and Species, are combined with a variety of interesting Information illustrative of the Habits, Instincts, and General Economy of the Animal Kingdom. With 900 Woodcuts. Fcp. 10s.

Maunder's Historical Treasury; comprising a General Introductory Outline of Universal History, Ancient and Modern, and a Series of Separate Histories of every principal Nation that exists; their Rise, Progress, and Present Condition, the Moral and Social Character of their respective Inhabitants, their Religion, Manners, and Customs, &c. Fcp. 8vo. 10s.

Maunder's Treasury of Geography, Physical, Historical, Descriptive, and Political; containing a succinct Account of Every Country in the World: Preceded by an Introductory Outline of the History of Geography; a Familiar Inquiry into the Varieties of Race and Language exhibited by different Nations; and a View of the Relations of Geography to Astronomy and the Physical Sciences. Completed by WILLIAM HUGHES, F.R.G.S. With 7 Maps and 16 Steel Plates. Fcp. 8vo. 10s.

Merivale (Miss). — Christian Records: A Short History of Apostolic Age. By L. A. MERIVALE. Fcp. 8vo. price 7s. 6d.

Merivale. — The Fall of the Roman Republic: A Short History of Last Century of the Commonwealth. By Rev. C. MERIVALE. 12mo. 7s. 6d.

Merivale. — A History of the Romans under the Empire. By the Rev. CHARLES MERIVALE, B.D., late Fellow of St. John's College, Cambridge. 8vo. with Maps.

VOLS. I and II. comprising the History to the Fall of *Julius Cæsar*. Second Edition. 28s.

VOL. III. to the Establishment of the Monarchy by *Augustus*. Second Edition......14s.

VOLS. IV. and V. from *Augustus* to *Claudius*, B.C. 27 to A.D. 54 ....32s.

VOL. VI. from the Reign of Nero, A.D. 54, to the Fall of Jerusalem, A.D. 70............16s.

Mildred Norman the Nazarene. By a WORKING MAN. Crown 8vo. 5s.

Miles.—The Horse's Foot and How to Keep it Sound. *Eighth Edition*; with an Appendix on Shoeing in general, and Hunters in particular. 12 Plates and 12 Woodcuts. By W. MILES, Esq. Imperial 8vo. 12s. 6d.

Miles's Plain Treatise on Horse-Shoeing. With Plates and Woodcuts. Second Edition. Post 8vo. 2s.

Milner's History of the Church of Christ. With Additions by the late Rev. ISAAC MILNER, D.D., F.R.S. A New Edition, revised, with additional Notes by the Rev. T. GRANTHAM, B.D. 4 vols. 8vo. 52s.

Minturn. — From New York to Delhi by way of Rio de Janeiro, Australia, and China. By ROBERT B. MINTURN, Jun. With coloured Route-Map of India. Post 8vo. 7s. 6d.

Mollhausen.—Diary of a Journey from the Mississippi to the Coasts of the Pacific, with a United States Government Expedition. By B. MÖLLHAUSEN, Topographical Draughtsman and Naturalist to the Expedition. With an Introduction by Baron HUMBOLDT; Map, coloured Illustrations, and Woodcuts. 2 vols. 8vo. 30s.

James Montgomery's Poetical Works: Collective Edition; with the Author's Autobiographical Prefaces, complete in One Volume; with Portrait and Vignette. Square crown 8vo. 10s. 6d. cloth; morocco, 21s.—Or, in 4 vols. fcp. 8vo. with Plates, 14s.

Moore.—The Power of the Soul over the Body, considered in relation to Health and Morals. By GEORGE MOORE, M.D. Fcp. 8vo. 6s.

Moore.—Man and his Motives. By George Moore, M.D. Fcp. 8vo. 6s.

Moore.—The Use of the Body in relation to the Mind. By G. Moore, M.D. Fcp. 8vo. 6s.

Moore.—Memoirs, Journal, and Correspondence of Thomas Moore. Edited by the Right Hon. Lord John Russell, M.P. With Portraits and Vignettes. 8 vols. post 8vo. £4. 4s.

Thomas Moore's Poetical Works: Comprising the Author's Autobiographical Prefaces, latest Corrections, and Notes. Various Editions of the separate Poems and complete Poetical Works, as follows:—

	s.	d.
LALLA ROOKH, 32mo. ruby type	1	0
LALLA ROOKH, 16mo. Vignette	2	6
LALLA ROOKH, square crown 8vo. Plates	15	0
LALLA ROOKH, fcp. 4to. with Woodcut Illustrations by Tenniel, in the press.		
IRISH MELODIES, 32mo. ruby type..	1	0
IRISH MELODIES, 16mo. Vignette ..	2	6
IRISH MELODIES, square crown 8vo. Plates	21	0
IRISH MELODIES, illustrated by Maclise, super-royal 8vo....	31	6
SONGS, BALLADS, and SACRED SONGS, 32mo. ruby type	2	6
SONGS, BALLADS, and SACRED SONGS, 16mo. Vignette	5	0
POETICAL WORKS, People's Edition, 10 Parts, each....	1	0
POETICAL WORKS, Cabinet Edition, 10 Vols. each....	3	6
POETICAL WORKS, Traveller's Edition, crown 8vo....	12	6
POETICAL WORKS, Library Edition, medium 8vo....	21	0
SELECTIONS, entitled "POETRY and PICTURES from THOMAS MOORE," fcp. 4to. with Wood Engravings....	21	0
MOORE'S EPICUREAN, 16mo. Vignette....	5	0

Editions printed with the Music.

	s.	d.
IRISH MELODIES, People's Edition, small 4to.	12	0
IRISH MELODIES, imperial 8vo. small music size	31	6
HARMONISED AIRS from IRISH MELODIES, imperial 8vo....	15	0
NATIONAL AIRS, People's Edition, 10 Nos. each	1	0
NATIONAL AIRS, imperial 8vo. small music size	31	6
SACRED SONGS and SONGS from SCRIPTURE, imperial 8vo.	16	0

No Edition of Thomas Moore's Poetical Works, or any separate Poem of Moore's, can be published complete except by Messrs. Longman and Co.

Morell. — Elements of Psychology: Part I., containing the Analysis of the Intellectual Powers. By J. D. Morell, M.A., One of Her Majesty's Inspectors of Schools. Post 8vo. 7s. 6d.

Morning Clouds. By the Author of *The Afternoon of Life.* Second Edition, revised throughout. Fcp. 8vo. 5s.

Morris (F. O.) — Anecdotes in Natural History. By the Rev. F. O. Morris, B.A., Rector of Nunburnholme, Yorkshire, Author of "History of the Nests and Eggs of British Birds," &c. Fcp. 8vo. [*Just ready.*

Morris (J.) — The Life and Martyrdom of St. Thomas Becket, Archbishop of Canterbury and Legate of the Holy See. By John Morris, Canon of Northampton. Post 8vo. 9s.

Morton.—The Resources of Estates: A Treatise on the Agricultural Improvement and General Management of Landed Property. By John Lockhart Morton, Civil and Agricultural Engineer; Author of Thirteen Highland and Agricultural Prize Essays. With 25 Lithographic Illustrations. Royal 8vo. 31s. 6d.

Moseley's Mechanical Principles of Engineering and Architecture. Second Edition, enlarged; with numerous Woodcuts. 8vo. 24s.

Memoirs and Letters of the late Colonel Armine Mountain, Aide-de-Camp to the Queen, and Adjutant-General of Her Majesty's Forces in India. Edited by Mrs. Mountain. Second Edition, Portrait. Fcp. 8vo. 6s.

Mure.—A Critical History of the Language and Literature of Ancient Greece. By William Mure, of Caldwell. Vols. I. to III. 8vo. price 36s.; Vol. IV. 15s.; and Vol. V. 18s.

Murray's Encyclopædia of Geography, comprising a complete Description of the Earth: Exhibiting its Relation to the Heavenly Bodies, its Physical Structure, the Natural History of each Country, and the Industry, Commerce, Political Institutions, and Civil and Social State of All Nations. Second Edition; with 82 Maps, and upwards of 1,000 other Woodcuts. 8vo. 60s.

Neale.—The Closing Scene; or, Christianity and Infidelity contrasted in the Last Hours of Remarkable Persons. By the Rev. Erskine Neale, M.A. 2 vols. fcp. 8vo. 6s. each.

Normanby (Marquis of).—A Year of Revolution. From a Journal kept in Paris in the Year 1848. By the MARQUIS OF NORMANBY, K.G. 2 vols. 8vo. 24s.

Ogilvie.—The Master-Builder's Plan; or, the Principles of Organic Architecture as indicated in the Typical Forms of Animals. By GEORGE OGILVIE, M.D. Post 8vo. with 72 Woodcuts, price 6s. 6d.

Oldacre.—The Last of the Old Squires. A Sketch. By CEDRIC OLDACRE, Esq., of Sax-Normanbury. Crown 8vo. 9s. 6d.

Osborn.—Quedah; or, Stray Leaves from a Journal in Malayan Waters. By Captain SHERARD OSBORN, R.N., C.B. With a coloured Chart and tinted Illustrations. Post 8vo. 10s. 6d.

Osborn.—The Discovery of the North-West Passage by H.M.S. *Investigator*, Captain R. M'CLURE, 1850-1854. Edited by Captain SHERARD OSBORN, C.B. Third Edition; with Portrait, Chart, and Illustrations. 8vo. 15s.

Professor Owen's Lectures on the Comparative Anatomy and Physiology of the Invertebrate Animals, delivered at the Royal College of Surgeons. Second Edition, with 235 Woodcuts. 8vo. 21s.

Professor Owen's Lectures on the Comparative Anatomy and Physiology of the Vertebrate Animals, delivered at the Royal College of Surgeons in 1844 and 1846. VOL. I. 8vo. 14s.

Memoirs of Admiral Parry, the Arctic Navigator. By his Son, the Rev. E. PARRY, M.A., Domestic Chaplain to the Bishop of London. Sixth Edition; with a Portrait and coloured Chart of the North-West Passage. Fcp. 8vo. 5s.

Pattison.—The Earth and the Word; or, Geology for Bible Students. By S. R. PATTISON, F.G.S. Fcp. 8vo. with coloured Map, 3s. 6d.

Peaks, Passes, and Glaciers: a Series of Excursions by Members of the Alpine Club. Edited by JOHN BALL, M.R.I.A., F.L.S., President of the Alpine Club. *Second Edition;* with numerous Maps, coloured Illustrations, and Engravings on Wood. Square crown 8vo. 21s.—The EIGHT SWISS MAPS, accompanied by a Table of the HEIGHTS of MOUNTAINS, may be had separately, price 3s. 6d.

Dr. Pereira's Elements of Materia Medica and Therapeutics. *Third Edition*, enlarged and improved from the Author's Materials by A. S. TAYLOR, M.D., and G. O. REES, M.D. Vol. I. 8vo. 28s.; Vol. II. Part I. 21s.; Vol. II. Part II. 26s.

Dr. Pereira's Lectures on Polar-ised Light, together with a Lecture on the Microscope. 2d Edition, enlarged from the Author's Materials by Rev. B. POWELL, M.A. Fcp. 8vo. Woodcuts, price 7s.

Perry.—The Franks, from their First Appearance in History to the Death of King Pepin. By WALTER C. PERRY, Barrister-at-Law. 8vo. 12s. 6d.

Peschel's Elements of Physics. Translated from the German, with Notes, by E. WEST. With Diagrams and Woodcuts. 3 vols. fcp. 8vo. 21s.

Phillips's Elementary Introduc-tion to Mineralogy. A New Edition, with extensive Alterations and Additions, by H. J. BROOKE, F.R.S., F.G.S.; and W. H. MILLER, M.A., F.G.S. With numerous Woodcuts. Post 8vo. 18s.

Phillips.—A Guide to Geology. By JOHN PHILLIPS, M.A., F.R.S., F.G.S., &c. Fourth Edition, corrected; with 4 Plates. Fcp. 8vo. 5s.

Piesse's Chymical, Natural, and Physical Magic, for the Instruction and Entertainment of Juveniles during the Holiday Vacation: with 30 Woodcuts and an Invisible Portrait of the Author. Fcp. 8vo. 3s. 6d.

Piesse's Art of Perfumery, and Methods of Obtaining the Odours of Plants; with Instructions for the Manufacture of Perfumes for the Handkerchief, Scented Powders, Odorous Vinegars, Dentifrices, Pomatums, Cosmétiques, Perfumed Soap, &c.; and an Appendix on the Colours of Flowers, Artificial Fruit Essences, &c. *Second Edition;* Woodcuts. Crown 8vo. 8s. 6d.

Pitt.—How to Brew Good Beer: A complete Guide to the Art of Brewing Ale, Bitter Ale, Table Ale, Brown Stout, Porter, and Table Beer. To which are added Practical Instructions for Making Malt. By JOHN PITT, Butler to Sir William R. P. Geary, Bart. Fcp. 8vo. 4s. 6d.

Porter.—History of the Knights of Malta, or the Order of the Hospital of St. John of Jerusalem. By Major WHITWORTH PORTER, Royal Engineers. With 5 Illustrations. 2 vols. 8vo. 24s.

Powell.—Essays on the Spirit of the Inductive Philosophy, the Unity of Worlds, and the Philosophy of Creation. By the Rev. BADEN POWELL, M.A., &c. Crown 8vo. Woodcuts, 12s. 6d.

Powell. — Christianity without Judaism: A Second Series of Essays on the Unity of Worlds and of Nature. By the Rev. BADEN POWELL, M.A., &c. Crown 8vo. 7s. 6d.

"This volume contains the pith of Professor Powell's argument urged often and powerfully against the Judaic spirit among Christians. ....Upon the theological part of Professor Powell's argument we offer no opinion; we simply desire to make known the nature of his book, and to secure for it the respect and attention it deserves." EXAMINER.

Powell.—The Order of Nature considered in reference to the Claims of Revelation: A Third Series of Essays on the Unity of Worlds and of Nature. By the Rev. BADEN POWELL, M.A. Crown 8vo. 12s.

Pycroft.—The Collegian's Guide; or, Recollections of College Days: Setting forth the Advantages and Temptations of a University Education. By the Rev. J. PYCROFT, B.A. *Second Edition.* Fcp. 8vo. 6s.

Pycroft's Course of English Reading; or, How and What to Read: Adapted to every taste and capacity. With Literary Anecdotes. Fcp. 8vo. 5s.

Pycroft's Cricket-Field; or, the Science and History of the Game of Cricket. Third Edition; Plates and Woodcuts. Fcp. 8vo. 5s.

Quatrefages (A. De).—Rambles of a Naturalist on the Coasts of France, Spain, and Sicily. By A. DE QUATREFAGES, Memb. Inst. Translated by E. C. OTTÉ. 2 vols. post 8vo. 15s.

Raikes (T.)—Portion of the Journal kept by THOMAS RAIKES, Esq., from 1831 to 1847: Comprising Reminiscences of Social and Political Life in London and Paris during that period. New Edition, complete in 2 vols. crown 8vo. price 12s.

Rich's Illustrated Companion to the Latin Dictionary and Greek Lexicon; Forming a Glossary of all the Words representing Visible Objects connected with the Arts, Manufactures, and Every-Day Life of the Ancients. With about 2,000 Woodcuts from the Antique. Post 8vo. 21s.

Richardson.—Fourteen Years' Experience of Cold Water: Its Uses and Abuses. By Captain M. RICHARDSON. Post 8vo. Woodcuts, 6s.

Horsemanship; or, the Art of Riding and Managing a Horse, adapted to the Guidance of Ladies and Gentlemen on the Road and in the Field: With Instructions for Breaking-in Colts and Young Horses. By Captain RICHARDSON, late of the 4th Light Dragoons. With 5 Plates. Square crown 8vo. 14s.

Riddle's Complete Latin-English and English-Latin Dictionary, for the use of Colleges and Schools. *New Edition*, revised and corrected. 8vo. 21s.

Riddle's Diamond Latin-English Dictionary. A Guide to the Meaning, Quality, and right Accentuation of Latin Classical Words. Royal 32mo. 4s.

Riddle's Copious and Critical Latin-English Lexicon, founded on the German-Latin Dictionaries of Dr. William Freund. Post 4to. 31s. 6d.

Rivers's Rose-Amateur's Guide; containing ample Descriptions of all the fine leading variety of Roses, regularly classed in their respective Families; their History and Mode of Culture. Sixth Edition. Fcp. 8vo. 3s. 6d.

Dr. E. Robinson's Greek and English Lexicon to the Greek Testament. A New Edition, revised and in great part re-written. 8vo. 18s.

Mr. Henry Rogers's Essays selected from Contributions to the *Edinburgh Review*. Second Edition, with Additions. 3 vols. fcp. 8vo. 21s.

Samuel Rogers's Recollections of Personal and Conversational Intercourse with Charles James Fox, Edmund Burke, Henry Grattan, Richard Porson, John Horne Tooke, Prince Talleyrand, Lord Erskine, Sir Walter Scott, Lord Grenville, *and* the Duke of Wellington. *Second Edition.* Fcp. 8vo. 5s.

Dr. Roget's Thesaurus of English Words and Phrases classified and arranged so as to facilitate the Expression of Ideas and assist in Literary Composition. Eighth Edition, revised and improved. Crown 8vo. 10s. 6d.

Ronalds's Fly-Fisher's Entomology: With coloured Representation of the Natural and Artificial Insects, and a few Observations and Instructions on Trout and Grayling Fishing. *Fifth Edition;* with 20 new-coloured Plates. 8vo. 14s.

Rowton's Debater: A Series of complete Debates, Outlines of Debates, and Questions for Discussion; with ample References to the best Sources of Information. Fcp. 8vo. 6s.

Dr. C. W. Russell's Life of Cardinal Mezzofanti: With an Introductory Memoir of eminent Linguists, Ancient and Modern. With Portrait and Facsimiles. 8vo. 12s.

Scherzer.—Travels in the Free States of Central America: Nicaragua, Honduras, and San Salvador. By Dr. CARL SCHERZER. 2 vols. post 8vo. 16s.

SchimmelPenninck (Mrs.) — Life of Mary Anne SchimmelPenninck. Edited by her relation, CHRISTIANA C. HANKIN. Third Edition, with Portrait. Post 8vo. 10s. 6d.

SchimmelPenninck's (Mrs.) Select Memoirs of Port Royal. *Fifth Edition*, revised, &c. by the Author's relation, CHRISTIANA C. HANKIN. 3 vols. post 8vo. 21s.

SchimmelPenninck's (Mrs.) Principles of Beauty; with an Essay on the Temperaments, and Thoughts on Grecian and Gothic Architecture. Edited by the Author's relation, C. C. HANKIN. Post 8vo. with coloured Illustrations, price 12s. 6d.

Dr. L. Schmitz's History of Greece, mainly based upon Bishop Thirlwall's History. *Fifth Edition*, with Nine new Supplementary Chapters on the Civilisation, Religion, Literature, and Arts of the Ancient Greeks, contributed by C. H. WATSON, M.A. Trin. Coll. Camb.; also a Map of Athens and 137 Woodcuts designed by G. Scharf, jun., F.S.A. 12mo. 7s. 6d.

Scoffern (Dr.)—Projectile Weapons of War and Explosive Compounds. By J. SCOFFERN, M.B. Lond., late Professor of Chemistry in the Aldersgate College of Medicine. *Fourth Edition.* Post 8vo. Woodcuts, 9s. 6d.

Senior.—Journal kept in Turkey and Greece in the Autumn of 1857 and the beginning of 1858. By NASSAU W. SENIOR, Esq. With 2 Maps and 2 Views. Post 8vo. 12s.

Sewell (Miss).—New Edition of the Tales and Stories of the Author of *Amy Herbert*, in 9 vols. crown 8vo. price £1. 10s. cloth; or each work complete in one volume, separately as follows:—

AMY HERBERT............2s. 6d.
GERTRUDE................2s. 6d.
The EARL'S DAUGHTER..2s. 6d.
The EXPERIENCE of LIFE..2s. 6d.
CLEVE HALL..............3s. 6d.
IVORS, or the Two COUSINS 3s. 6d.
KATHARINE ASHTON....3s. 6d.
MARGARET PERCIVAL ..5s. 0d.
LANETON PARSONAGE ..4s. 6d.

Also by the Author of Amy Herbert.

Ursula: A Tale of English Country Life. 2 vols. fcp. 8vo. 12s.

History of the Early Church: from the First Preaching of the Gospel to the Council of Nicea. 18mo. 4s. 6d.

Self-Examination before Confirmation: With Devotions and Directions for Confirmation-Day. 32mo. 1s. 6d.

Readings for a Month preparatory to Confirmation: Compiled from the Works of Writers of the Early and of the English Church. Fcp. 8vo. 4s.

Readings for every Day in Lent: Compiled from the Writings of Bishop JEREMY TAYLOR. Fcp. 8vo. 5s.

Bowdler's Family Shakspeare: In which nothing is *added* to the Original Text; but those words and expressions are *omitted* which cannot with propriety be read aloud. Illustrated with 36 Woodcut Vignettes. The *Library Edition*, in One Volume, medium 8vo. price 21s.; a *Pocket Edition*, in 6 vols. fcp. 8vo. price 5s. each.

Sharp's New British Gazetteer, or Topographical Dictionary of the British Islands and narrow Seas: Comprising concise Descriptions of about 60,000 Places, Seats, Natural Features, and Objects of Note, founded on the best authorities. 2 vols. 8vo. £2. 16s.

Short Whist; its Rise, Progress, and Laws: With Observations to make any one a Whist-Player. Containing also the Laws of Piquet, Cassino, Ecarté, Cribbage, Backgammon. By Major A. New Edition; with Precepts for Tyros, by Mrs. B. Fcp. 8vo. 3s.

Simpson.—Handbook of Dining; or, How to Dine, theoretically, philosophically, and historically considered: Based chiefly upon the *Physiologie du Goût* of Brillat-Savarin. By LEONARD FRANCIS SIMPSON, M.R.S.L. Fcp. 8vo. 5s.

Sinclair. — The Journey of Life. By CATHERINE SINCLAIR, Author of *The Business of Life.* Fcp. 8vo. 5s.

Sir Roger De Coverley. From the Spectator. With Notes and Illustrations, by W. HENRY WILLS; and 12 Wood Engravings from Designs by F. TAYLER. Crown 8vo. 10s. 6d.; or 21s. in morocco by Hayday.

The Sketches: Three Tales. By the Authors of *Amy Herbert, The Old Man's Home,* and *Hawkstone.* Fcp. 8vo. price 4s. 6d.

Smee's Elements of Electro-Metallurgy. Third Edition, revised; with Electrotypes and numerous Woodcuts. Post 8vo. 10s. 6d.

Smith (G.)—History of Wes-leyan Methodism. By GEORGE SMITH, F.A.S., Author of *Sacred Annals,* &c. VOL. I. *Wesley and his Times;* VOL. II. *The Middle Age of Methodism,* from 1791 to 1816. Crown 8vo. 10s. 6d. each.

Smith (J.)—The Voyage and Shipwreck of St. Paul: With Dissertations on the Life and Writings of St. Luke, and the Ships and Navigation of the Ancients. By JAMES SMITH, F.R.S. With Charts, Views, and Woodcuts. Crown 8vo. 8s. 6d.

A Memoir of the Rev. Sydney Smith. By his Daughter, LADY HOLLAND. With a Selection from his Letters, edited by Mrs. AUSTIN. *New Edition.* 2 vols. 8vo. 28s.

The Rev. Sydney Smith's Mis-cellaneous Works: Including his Contributions to The Edinburgh Review. Four Editions:—

1. A LIBRARY EDITION (the *Fourth*), in 3 vols. 8vo. with Portrait, 36s.
2. Complete in ONE VOLUME, with Portrait and Vignette. Square crown, 8vo. 21s. cloth; or 30s. bound in calf.
3. Another NEW EDITION, in 3 vols. fcp. 8vo. 21s.
4. The People's Edition, in 2 vols. crown 8vo. price 8s. cloth.

The Rev. Sydney Smith's Ele-mentary Sketches of Moral Philosophy, delivered at the Royal Institution in the Years 1804 to 1806. Fcp. 8vo. 7s.

Snow.—Two Years' Cruise off Tierra del Fuego, the Falkland Islands, Patagonia, and in the River Plate: A Narrative of Life in the Southern Seas. By W. PARKER SNOW, late Commander of the Mission Yacht *Allen Gardiner.* With Charts and Illustrations. 2 vols. post 8vo. 24s.

Robert Southey's Complete Poet-ical Works; containing all the Author's last Introductions and Notes. The *Library Edition,* complete in One Volume, with Portraits and Vignette. Medium 8vo. 21s. cloth; 42s. bound in morocco.—Also, the *First collected Edition,* in 10 vols. fcp. 8vo. with Portrait and 19 Vignettes, price 35s.

Southey's Doctor, complete in One Volume. Edited by the Rev. J. W. WARTER, B.D. With Portrait, Vignette, Bust, and coloured Plate. Square crown 8vo. 21s.

Southey's Life of Wesley; and Rise and Progress of Methodism. Fourth Edition, edited by Rev. C. C. SOUTHEY, M.A. 2 vols. crown 8vo. 12s.

Spencer.—Essays, Scientific, Po-litical, and Speculative. By HERBERT SPENCER, Author of *Social Statics.* Reprinted chiefly from Quarterly Reviews. 8vo. 12s. cloth.

Spencer.—The Principles of Psychology. By HERBERT SPENCER, Author of *Social Statics.* 8vo. 16s.

Stephen.—Lectures on the His-tory of France. By the Right Hon. Sir JAMES STEPHEN, K.C.B., LL.D. Third Edition. 2 vols. 8vo. 24s.

Stephen.—Essays in Ecclesiastical Biography; from The Edinburgh Review. By the Right Hon. Sir JAMES STEPHEN, K.C.B., LL.D. Third Edition. 2 vols. 8vo. 24s.

Stonehenge.—The Dog in Health and Disease: Comprising the various Modes of Breaking and using him for Hunting, Coursing, Shooting, &c.; and including the Points or Characteristics of Toy Dogs. By STONEHENGE. With about 70 Illustrations engraved on Wood. Square crown 8vo. price 15s. half-bound.

Stonehenge's Work on the Greyhound: Being a Treatise on the Art of Breeding, Rearing, and Training Greyhounds for Public Running; their Diseases and Treatment: Containing also Rules for the Management of Coursing Meetings, and for the Decision of Courses. With Frontispiece and Woodcuts. Square crown 8vo. 21s.

Stow's Training System, Moral Training School, and Normal Seminary for preparing Schoolmasters and Governesses. Eleventh Edition; Plates and Woodcuts. Post 8vo. 6s. 6d.

Strickland.—Lives of the Queens of England. By AGNES STRICKLAND. Dedicated, by express permission, to Her Majesty. Embellished with Portraits of every Queen, engraved from the most authentic sources. Complete in 8 vols. post 8vo. 7s. 6d. each.

Symonds.—Memoirs of the Life and Services of Rear-Admiral Sir William Symonds, late Surveyor of the Navy. Edited by J. A. SHARP. 8vo. with Illustrations, price 21s.

Taylor.—Loyola: and Jesuitism in its Rudiments. By ISAAC TAYLOR. Post 8vo. Medallion, 10s. 6d.

Taylor.—Wesley and Methodism. By ISAAC TAYLOR. Post 8vo. Portrait, 10s. 6d.

Tennent.—Ceylon: An Account of the Island, Physical, Historical, and Topographical: with Copious Notices of its Natural History, Antiquities, and Productions. Illustrated by 7 Maps, 17 Plans and Charts, and 101 Engravings on Wood. By Sir J. EMERSON TENNENT, K.C.S., LL.D., &c. 2 vols. 8vo. price 50s.

Bishop Thirlwall's History of Greece. Library Edition; with Maps. 8 vols. 8vo. £3.—An Edition in 8 vols. fcp. 8vo. with Vignette Titles, 28s.

Thomson's Seasons. Edited by BOLTON CORNEY, Esq. Illustrated with 77 fine Wood Engravings from Designs by Members of the Etching Club. Square crown 8vo. 21s. cloth; or 36s. bound in morocco.

Thomson (the Rev. Dr.)—An Outline of the necessary Laws of Thought: A Treatise on Pure and Applied Logic. By WILLIAM THOMSON, D.D. New Edition. Fcp. 8vo. 7s. 6d.

Thomson's Tables of Interest, at Three, Four, Four-and-a-Half, and Five per Cent., from One Pound to Ten Thousand, and from 1 to 365 Days, in a regular progression of single Days; with Interest at all the above Rates, from One to Twelve Months, and from One to Ten Years. Also, numerous other Tables of Exchange, Time, and Discounts. The Seventeenth Edition, thoroughly revised and stereotyped. 12mo. 8s. 6d.

The Thumb Bible; or, Verbum Sempiternum. By J. TAYLOR. Being an Epitome of the Old and New Testaments in English Verse. Reprinted from the Edition of 1693. 64mo. 1s. 6d.

Todd (Dr.)—The Cyclopædia of Anatomy and Physiology. Edited by ROBERT B. TODD, M.D., F.R.S., &c., Physician to King's College Hospital; late Professor of General and Morbid Anatomy in King's College, London. Now complete in 5 vols. 8vo. pp. 5,350, illustrated with 2,853 Woodcuts, price £6. 6s. cloth.

Tooke.—History of Prices, and of the State of the Circulation, during the Nine Years from 1848 to 1856 inclusive. Forming Vols. V. and VI. of Tooke's *History of Prices;* and comprising a copious Index to the whole work. By THOMAS TOOKE, F.R.S. and WILLIAM NEWMARCH. 2 vols. 8vo. 52s. 6d.

Trevelyan (Sir C.)—Original Papers illustrating the History of the Application of the Roman Alphabet to the Languages of India. Edited by MONIER WILLIAMS, M.A., late Professor of Sanskrit in the East-India College, Haileybury. 8vo. with Map, price 12s.

The Traveller's Library: A Collection of original Works well adapted for *Travellers* and *Emigrants*, for *School-room Libraries*, the *Libraries of Mechanics' Institutions*, *Young Men's Libraries*, the *Libraries of Ships*, and similar purposes. The separate volumes are suited for *School Prizes*, *Presents to Young People*, and for general instruction and entertainment. The Series comprises fourteen of the most popular of Lord Macaulay's *Essays*, and his *Speeches* on Parliamentary Reform. The department of Travels contains some account of eight of the principal countries of Europe, as well as travels in four districts of Africa, in four of America, and in three of Asia. Madame Pfeiffer's *First Journey round the World* is included; and a general account of the *Australian Colonies*. In Biography and History will be found Lord Macaulay's Biographical Sketches of *Warren Hastings*, *Clive*, *Pitt*, *Walpole*, *Bacon*, and others; besides Memoirs of *Wellington*, *Turenne*, *F. Arago*, &c.; an Essay on the Life and Genius of *Thomas Fuller*, with Selections from his Writings, by Mr. Henry Rogers; and a history of the *Leipsic Campaign*, by Mr. Gleig,—which is the only separate account of this remarkable campaign. Works of Fiction did not come within the plan of the TRAVELLER'S LIBRARY; but the *Confessions of a Working Man*, by Souvestre, which is indeed a fiction founded on fact, has been included, and has been read with unusual interest by many of the working classes, for whose use it is especially recommended. Dumas's story of the *Maitre-d'Armes*, though in form a work of fiction, gives a striking picture of an episode in the history of Russia. Amongst the works on Science and Natural Philosophy, a general view of Creation is embodied in Dr. Kemp's *Natural History of Creation*; and in his *Indications of Instinct* remarkable facts in natural history are collected. Dr. Wilson has contributed a popular account of the *Electric Telegraph*. In the volumes on the *Coal-Fields*, and on the Tin and other Mining Districts of *Cornwall*, is given an account of the mineral wealth of England, the habits and manners of the miners, and the scenery of the surrounding country. It only remains to add, that among the Miscellaneous Works are a Selection of the best Writings of the Rev. Sydney Smith; Lord Carlisle's *Lectures and Addresses*; an account of *Mormonism*, by the Rev. W. J. Conybeare; an exposition of *Railway* management and mismanagement by Mr. Herbert Spencer; an account of the Origin and Practice of *Printing*, by Mr. Stark; and an account of *London*, by Mr. M'Culloch.—To be had, in *complete Sets only*, at £5. 5s. per Set, bound in cloth and lettered.

☞ *The Traveller's Library* may also be had as originally issued in 102 parts, 1s. each, forming 50 vols. 2s. 6d. each; or any separate parts or volumes.

Trollope.—The Warden, a Novel. By ANTHONY TROLLOPE. New and cheaper Edition. Crown 8vo. 2s. 6d.

Trollope's Barchester Towers, a Sequel to *The Warden*. New and cheaper Edition, complete in One Volume. Crown 8vo. 5s.

Sharon Turner's History of the Anglo-Saxons, from the Earliest Period to the Norman Conquest. 3 vols. 36s.

Dr. Turton's Manual of the Land and Fresh-Water Shells of Great Britain: With Figures of each of the kinds. New Edition, with Additions by Dr. J. E. GRAY, F.R.S., &c., Keeper of the Zoological Collection in the British Museum. Crown 8vo. with 12 coloured Plates, price 15s. cloth.

Dr. Ure's Dictionary of Arts, Manufactures, and Mines: Containing a clear Exposition of their Principles and Practice. New Edition, chiefly rewritten and greatly enlarged; with nearly 2,000 Woodcuts. Edited by ROBERT HUNT, F.R.S., F.S.S., Keeper of Mining Records. In course of publication in 14 Parts, price 5s. each, forming 3 vols. 8vo.

Uwins. — Memoir and Correspondence of THOMAS UWINS, R.A., late keeper of the Royal Galleries and of the National Gallery, &c. Edited by Mrs. UWINS. 2 vols. post 8vo. 18s.

Van der Hoeven's Handbook of Zoology. Translated by the Rev. WILLIAM CLARK, M.D., F.R.S., Professor of Anatomy in the University of Cambridge. 2 vols. 8vo. with 24 Plates of Figures, price 60s. cloth; or separately, VOL. I. *Invertebrata*, 30s., and VOL. II. *Vertebrata*, 30s.

Vehse.—Memoirs of the Court, Aristocracy, and Diplomacy of Austria. By Dr. E. VEHSE. Translated from the German by FRANZ DEMMLER. 2 vols. post 8vo. 21s.

Von Tempsky.—Mitla; or, Incidents and Personal Adventures on a Journey in Mexico, Guatemala, and Salvador in the Years 1853 to 1855. By G. F. VON TEMPSKY. With numerous Illustrations. 8vo. 18s.

Wade. — England's Greatness: Its Rise and Progress in Government, Laws, Religion, and Social Life; Agriculture, Commerce, and Manufactures; Science, Literature and Arts, from the Earliest Period to the Peace of Paris. By JOHN WADE, Author of the *Cabinet Lawyer*, &c. Post 8vo. 10s. 6d.

Wanderings in the Land of Ham. By a DAUGHTER of JAPHET. Post 8vo. 8s. 6d.

Waterton.—Essays on Natural History, chiefly Ornithology. By C. WATERTON, Esq. With the Autobiography of the Author, and Views of Walton Hall. 2 vols. fcp. 8vo. 5s. each.

Waterton's Essays on Natural History. THIRD SERIES; with a Continuation of the Autobiography, and a Portrait of the Author. Fcp. 8vo. 6s.

Watson. — Cybele Britannica; or, British Plants and their Geographical Relations. By HEWETT COTTRELL WATSON. 4 vols. 8vo. 42s. cloth; or each vol. separately, price 10s. 6d.

Webb. — Celestial Objects for Common Telescopes. By the Rev. T. W. WEBB, M.A., F.R.A.S., Incumbent of Hardwick, Herefordshire. With Woodcuts, and Map of the Moon 12 inches in diameter engraved on Steel, 16mo. 7s.

Webster and Parkes's Encyclopædia of Domestic Economy; comprising such subjects as are most immediately connected with Housekeeping: viz. The Construction of Domestic Edifices, with the Modes of Warming, Ventilating, and Lighting them—A description of the various Articles of Furniture, with the Nature of their Materials—Duties of Servants—&c. With nearly 1,000 Woodcuts. 8vo. 50s.

Weld. — The Pyrenees, West and East. By CHARLES RICHARD WELD, Barrister-at-Law. With 8 Illustrations in Chromo-xylography from Drawings by the Author. Post 8vo. 12s. 6d.

Weld's Vacation Tour in the United States and Canada. 10s. 6d.

Weld's Vacations in Ireland. Post 8vo. 10s. 6d.

Willich's Popular Tables for ascertaining the Value of Lifehold, Leasehold, and Church Property, Renewal Fines, &c. With numerous additional Tables—Chemical, Astronomical, Trigonometrical, Common and Hyperbolic Logarithms; Constants, Squares, Cubes, Roots, Reciprocals, &c. Fourth Edition. Post 8vo. 10s.

Wilmot's Abridgment of Blackstone's Commentaries on the Laws of England, in a series of Letters from a Father to his Daughter. 12mo. 6s. 6d.

Wilson's Bryologia Britannica: Containing the Mosses of Great Britain and Ireland systematically arranged and described according to the Method of *Bruch* and *Schimper*; with 61 illustrative Plates. Being a New Edition, enlarged and altered, of the *Muscologia Britannica* of Messrs. Hooker and Taylor. 8vo. 42s.; or, with the Plates coloured, price £4. 4s.

Yonge.—A New English-Greek Lexicon: Containing all the Greek Words used by Writers of good authority. By C. D. YONGE, B.A. *Second Edition*, revised. Post 4to. 21s.

Yonge's New Latin Gradus: Containing Every Word used by the Poets of good authority. For the use of Eton, Westminster, Winchester, Harrow, and Rugby Schools; King's College, London; and Marlborough College. *Sixth Edition.* Post 8vo. 9s.; or, with APPENDIX of *Epithets*, 12s.

Youatt's Work on the Horse: With a Treatise on Draught. New Edition, revised and enlarged by E. N. GABRIEL, M.R.C.S., C.V.S., Secretary to the Royal College of Veterinary Surgeons. With numerous Woodcut Illustrations, chiefly from designs by W. Harvey. 8vo. price 10s. 6d. cloth.

Youatt.—The Dog. By William Youatt. A New Edition; with numerous Engravings, from Designs by W. Harvey. 8vo. 6s.

Young.—The Christ of History: An Argument grounded in the Facts of His Life on Earth. By JOHN YOUNG, LL.D. Second Edition. Post 8vo. 7s. 6d.

Young.—The Mystery; or, Evil and God. By JOHN YOUNG, LL.D. Post 8vo. 7s. 6d.

Zumpt's Grammar of the Latin Language. Translated and adapted for the use of English Students by Dr. L. SCHMITZ, F.R.S.E.: With numerous Additions and Corrections by the Author and Translator. 8vo. 14s.

[*September* 1859.

LONDON: PRINTED BY SPOTTISWOODE & CO. NEW-STREET SQUARE.